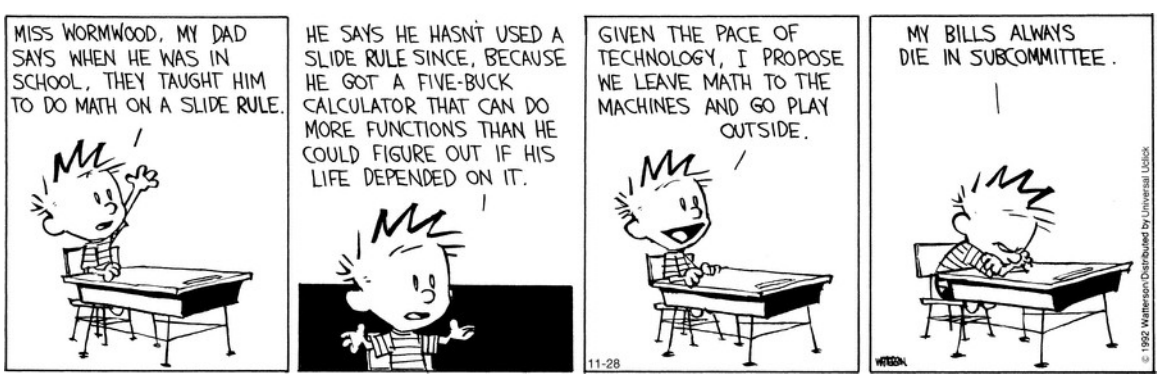
MATH 218D-1
MIDTERM EXAMINATION 1

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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Problem 1.

[20 points]

Consider the matrix

$$A = \begin{pmatrix} 0 & 2 & -2 & -2 \\ 2 & 0 & 8 & 0 \\ -3 & -1 & -3 & 6 \\ 6 & 0 & 12 & -6 \end{pmatrix}.$$

a) Perform Gaussian elimination with maximal partial pivoting to obtain a $PA = LU$ decomposition of A. You should end up with

$$U = \begin{pmatrix} 6 & 0 & 12 & -6 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Please write the row operations you performed. (You can continue your work on the back of this sheet.)

$$L = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix} \qquad P = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

b) Briefly explain the reason one might want to always choose the largest pivot in absolute value.

[Scratch work for Problem 1]

Problem 2. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -1 & -3 & 4 \\ -2 & -6 & 9 \end{pmatrix}.$$

a) Compute A^{-1}. Please write the row operations you performed.

$$A^{-1} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Express A^{-1} as a product of elementary matrices. (Your answer will be a product of matrices with numbers in them, as opposed to row operations.)

$$A^{-1} =$$

c) Solve $Ax = b$, where $b = (b_1, b_2, b_3)$ is an unknown vector. (Your answer will be a formula in b_1, b_2, b_3.)

$$x = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

[Scratch work for Problem 2]

Problem 3.

[20 points]

Consider the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 2 \\ x_2 + x_3 + 2x_4 &= 1. \end{aligned}$$

a) Express the solution set as a translate of a span:

$$\text{solution set} = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} + \operatorname{Span}\left\{ \qquad\qquad\qquad \right\}$$

b) The solution set is a (circle one) $\begin{pmatrix}\text{point}\\ \text{line}\\ \text{plane}\\ \text{space}\end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\square}$.

c) The solution set of $Ax = 0$ has dimension $\square$.

d) Describe $\operatorname{Span}\left\{\binom{1}{0}, \binom{2}{1}, \binom{-1}{1}, \binom{-1}{2}\right\}$ geometrically:

it is a (circle one) $\begin{pmatrix}\text{point}\\ \text{line}\\ \text{plane}\\ \text{space}\end{pmatrix}$ in (fill in the blank) $\mathbf{R}^{\square}$.

e) Find numbers b_1, b_2 making the system

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= b_1 \\ x_2 + x_3 + 2x_4 &= b_2 \end{aligned}$$

inconsistent. If no such numbers exist, explain why.

$b_1 = \square \qquad b_2 = \square$

[Scratch work for Problem 3]

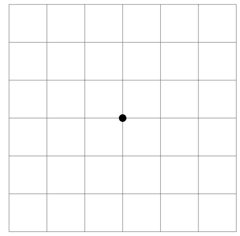
Problem 4.

[12 points]

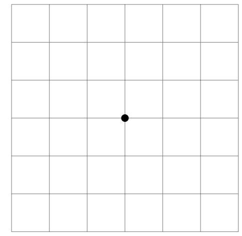
Give examples of 2×2 matrices A, B, C with ranks 0, 1, and 2, respectively. Draw pictures of the solution set of $Ax = 0$ and the span of the columns of A, and likewise for B and C. (Recall that the *rank* of a matrix is the number of pivots.) Be precise!

a) Rank 0: $A = \begin{pmatrix} & \\ & \end{pmatrix}$

$Ax = 0$

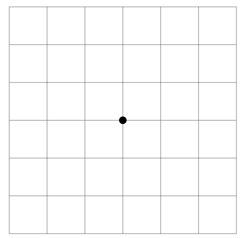


Span{cols of A}

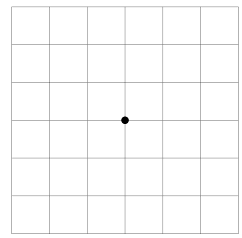


b) Rank 1: $B = \begin{pmatrix} & \\ & \end{pmatrix}$

$Bx = 0$

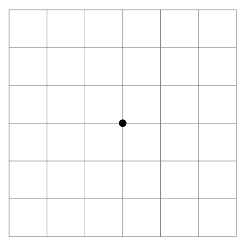


Span{cols of B}

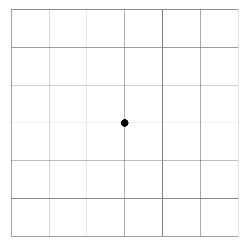


c) Rank 2: $C = \begin{pmatrix} & \\ & \end{pmatrix}$

$Cx = 0$



Span{cols of C}



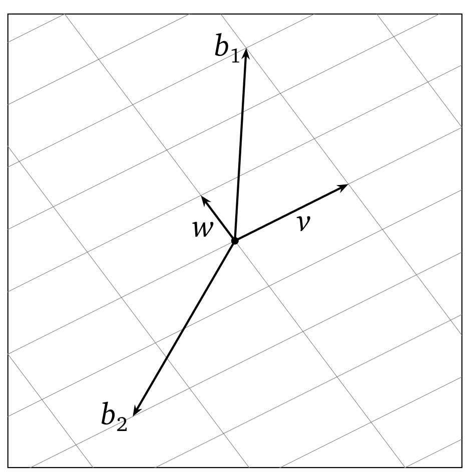
Problem 5.

[10 points]

A certain 2×2 matrix A has columns v and w, pictured below. Solve the equations $Ax_1 = b_1$ and $Ax_2 = b_2$, where b_1 and b_2 are the vectors in the picture.

$x_1 = \begin{pmatrix} \\ \end{pmatrix}$

$x_2 = \begin{pmatrix} \\ \end{pmatrix}$



[Scratch work for Problems 4 and 5]

Problem 6.

[16 points]

Short-answer questions: you do not need to justify your answers.

a) Suppose that A is a 4×2 matrix such that the solution set of $A\binom{x}{y} = 0$ is the line $y = x$. Let b be a *nonzero* vector in $\mathbf{R}^4$. Which of the following are definitely *not* the solution set of $Ax = b$? (Circle all that apply.)

The line $y = x$. The y-axis. The line $y = x + 1$.

The point $(1, 2, 3, 0)$. The empty set.

b) Consider the following plane in $\mathbf{R}^3$:

$$P = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

Find two *other* vectors that span P. Your answer cannot contain a scalar multiple of $(1, 0, -1)$ or $(1, -1, 0)$.

$$P = \operatorname{Span}\left\{ \begin{pmatrix} \\ \\ \\ \end{pmatrix}, \begin{pmatrix} \\ \\ \\ \end{pmatrix} \right\}$$

c) Find three vectors $u, v, w \in \mathbf{R}^3$ such that $\operatorname{Span}\{u, v, w\}$ is a *plane*, but such that $w \notin \operatorname{Span}\{u, v\}$.

$$u = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \qquad v = \begin{pmatrix} \\ \\ \\ \end{pmatrix} \qquad w = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

d) A *nonzero* 2×3 matrix A has the property that $Ax = \binom{1}{1}$ is inconsistent. The span of the columns of A is a

point line plane space.

[Scratch work for Problem 6]