MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

Consider the plane

$$V = \{(x,y,z) \,:\, x - y + 2z = 0\}.$$

a) Find a basis for V.

b) Find an orthogonal basis for V.

c) Use the projection formula and your answer to part **b)** to compute the orthogonal projection b_V of the vector $b = (1,1,-3)$ onto V.

d) Find a basis for $V^\perp$.

e) Find an orthogonal basis of $\mathbf{R}^3$ containing the basis vectors you found in **b)**.

Solution.

a) There are many answers. If you find the solutions of $x - y + 2z = 0$ in parametric vector form, you get

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix} \right\}.$$

b) Running Gram–Schmidt on the above vectors gives

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \right\}.$$

c)

$$b_V = \frac{\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\0\end{pmatrix}}\begin{pmatrix}1\\1\\0\end{pmatrix} + \frac{\begin{pmatrix}1\\1\\-3\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}{\begin{pmatrix}-1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\1\\1\end{pmatrix}}\begin{pmatrix}-1\\1\\1\end{pmatrix} = \begin{pmatrix}2\\0\\-1\end{pmatrix}.$$

d) Since $V = \mathrm{Nul}\begin{pmatrix}1 & -1 & 2\end{pmatrix}$, the orthogonal complement $V^\perp$ is the row space of $\begin{pmatrix}1 & -1 & 2\end{pmatrix}$:

$$\left\{ \begin{pmatrix}1\\-1\\2\end{pmatrix} \right\}.$$

e) We just add the vectors in **b)** and **d)**:

$$\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\2\end{pmatrix} \right\}.$$

(You could also notice that $b - b_V = (-1,1,-2)$ spans $V^\perp$.)

Problem 2. [20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}.$$

a) Find the QR decomposition of A. You should get $R = \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$.

b) Solve $R\widehat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$ to find the least-squares solution of $Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$.

c) Compute the matrix P_V for projection onto $V = \mathrm{Col}(A)$.

Solution.

a)

$$Q = \begin{pmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{pmatrix}$$

b)

$$\widehat{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

c)

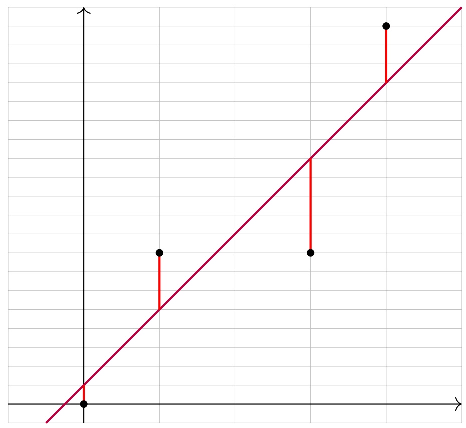
$$P_V = QQ^T = \frac{1}{10} \begin{pmatrix} 7 & -2 & 3 & 2 & 2 \\ -2 & 2 & 2 & -2 & -2 \\ 3 & 2 & 7 & -2 & -2 \\ 2 & -2 & -2 & 7 & -3 \\ 2 & -2 & -2 & -3 & 7 \end{pmatrix}$$

Problem 3. [15 points]

Consider the data points

$$b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

drawn below.



a) Find the matrix A such that the least-squares solution $\widehat{x} = (C, D)$ of

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

gives the coefficients of the best-fit line $y = Cx + D$.

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

c) Compute the minimized vector $b_{V^\perp}$. What does $b_{V^\perp}$ represent in the original best-fit problem? (Here $V = \mathrm{Col}(A)$.)

d) What is the best-fit line among all lines *passing through the origin*?

Solution.

a) $$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}$$

b) $$\widehat{x} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \quad \rightsquigarrow \quad y = 4x + 1$$

c) The minimized vector is

$$b_{V^\perp} = b - A\widehat{x} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ 13 \\ 17 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -5 \\ 3 \end{pmatrix}.$$

This is the vector of vertical distances from the data points to the graph of the best-fit line, drawn in red in the picture.

d) Using $y = Cx$ means solving the least-squares problem

$$\begin{pmatrix} 0 \\ 1 \\ 3 \\ 4 \end{pmatrix} C = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} \quad \rightsquigarrow \quad C = \frac{56}{13}.$$

The best-fit line is $y = \frac{56}{13}x$.

Problem 4.

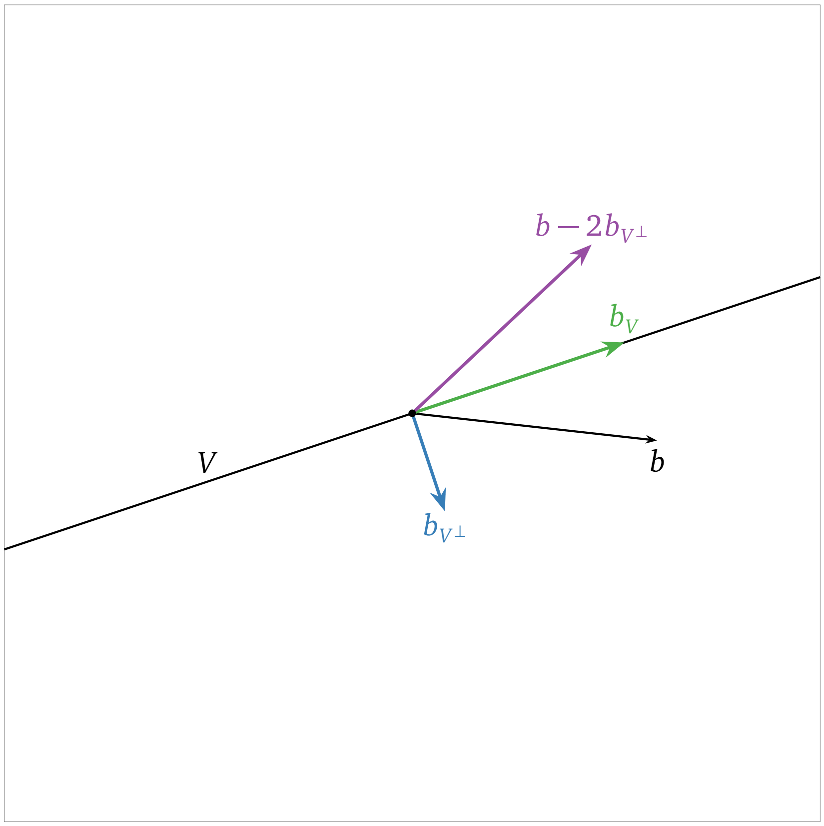
[12 points]

A line V and a vector b are drawn below. Draw *and label*:

a) The orthogonal projection b_V.

b) The projection onto the orthogonal complement $b_{V^\perp}$.

c) The vector $b - 2b_{V^\perp}$.



Problem 5. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) Nul$\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix}$

b) Col$\begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix}$

c) The subspace of all vectors in $\mathbf{R}^4$ whose entries sum to zero.

d) The line $\{(t, 2t, 3t) : t \in \mathbf{R}\}$.

e) $\mathbf{R}^3$

Solution.

These are the bases you would obtain if you did the problem the same way I did.

$$\textbf{a)}\ \left\{\begin{pmatrix}1\\0\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\4\\1\end{pmatrix}\right\} \qquad \textbf{b)}\ \left\{\begin{pmatrix}-3\\-6\\1\\0\end{pmatrix}, \begin{pmatrix}-4\\-9\\0\\1\end{pmatrix}\right\} \qquad \textbf{c)}\ \left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\right\}$$

$$\textbf{d)}\ \left\{\begin{pmatrix}-2\\1\\0\end{pmatrix}, \begin{pmatrix}-3\\0\\1\end{pmatrix}\right\} \qquad \textbf{e)}\ \{\}$$

Problem 6. [16 points]

a) Let A be an $m \times n$ matrix of rank r. Which of the following statements are equivalent to "A has full row rank"?

(1) $\mathrm{Nul}(A^T) = \{0\}$
(2) $n = r$
(3) $\mathrm{Col}(A) = \mathbf{R}^m$
(4) A has linearly independent columns
(5) A has a pivot in every row
(6) A is invertible
(7) $Ax = b$ is consistent for every vector b

b) Explain why the projection matrix P_V onto a subspace V can be written as QQ^T for some matrix Q with orthonormal columns. (What is Q in terms of V?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly *de*pendent, but v_3 is not in $\mathrm{Span}\{v_1, v_2\}$. Be sure to label which is v_3.

d) Give an example of a 4×4 matrix A such that $\mathrm{Nul}(A) = \mathrm{Row}(A)$, or explain why no such matrix exists.

Solution.

a) (1),(3),(5),(7)

b) Choose an orthonormal basis for V, then let Q be the matrix with those columns.

c) There are many answers. One is

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

d) This is impossible since $\mathrm{Nul}(A) = \mathrm{Row}(A)^\perp$.