MATH 218D-1
PRACTICE MIDTERM EXAMINATION 2

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

> This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1.

[20 points]

Consider the plane

$$V = \{(x, y, z) \; : \; x - y + 2z = 0\}.$$

a) Find a basis for V.

$$\left\{ \qquad \qquad \right\}$$

b) Find an orthogonal basis for V.

$$\left\{ \qquad \qquad \right\}$$

c) Use the projection formula and your answer to part **b)** to compute the orthogonal projection b_V of the vector $b = (1, 1, -3)$ onto V.

$$b_V = \left(\qquad \right)$$

d) Find a basis for $V^{\perp}$.

$$\left\{ \qquad \right\}$$

e) Find an orthogonal basis of $\mathbf{R}^3$ containing the basis vectors you found in **b)**.

$$\left\{ \qquad \qquad \qquad \right\}$$

Problem 2.

[20 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{pmatrix}.$$

a) Find the QR decomposition of A. You should get $R = \begin{pmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{pmatrix}$.

$$Q = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Solve $R\widehat{x} = Q^T \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$ to find the least-squares solution of $Ax = \begin{pmatrix} 2 \\ -2 \\ 4 \\ -3 \\ 3 \end{pmatrix}$.

$$\widehat{x} = \begin{pmatrix} \quad \end{pmatrix}$$

c) Compute the matrix P_V for projection onto $V = \mathrm{Col}(A)$.

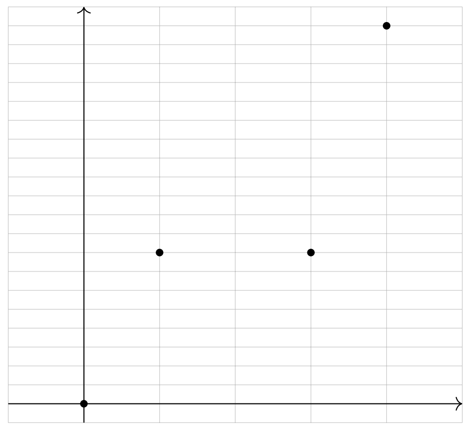
$$P_V = \begin{pmatrix} \quad \end{pmatrix}$$

Problem 3. [15 points]

Consider the data points

$$b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad b_1 = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \quad b_2 = \begin{pmatrix} 3 \\ 8 \end{pmatrix} \quad b_3 = \begin{pmatrix} 4 \\ 20 \end{pmatrix}$$

drawn below.



a) Find the matrix A such that the least-squares solution $\widehat{x} = (C, D)$ of

$$A\begin{pmatrix} C \\ D \end{pmatrix} = b = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}$$

gives the coefficients of the best-fit line $y = Cx + D$.

$$A = \begin{pmatrix} \quad \end{pmatrix}$$

b) Find the equation of the best-fit line by computing the least-squares solution of the above equation. Graph this line in the above grid.

$$y = \boxed{}\, x + \boxed{}$$

c) Compute the minimized vector $b_{V^\perp}$. What does $b_{V^\perp}$ represent in the original best-fit problem? (Here $V = \text{Col}(A)$.)

$$b_{V^\perp} = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

d) What is the best-fit line among all lines *passing through the origin*?

$$y = \boxed{}\,x$$

Problem 4.

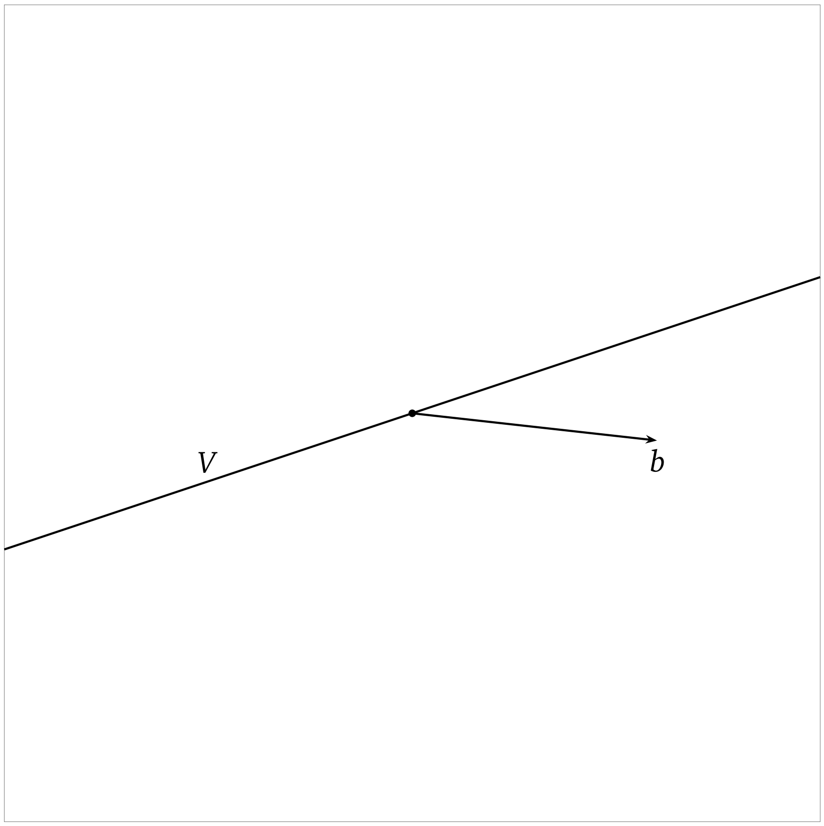
[12 points]

A line V and a vector b are drawn below. Draw *and label*:

a) The orthogonal projection b_V.

b) The projection onto the orthogonal complement $b_{V^\perp}$.

c) The vector $b - 2b_{V^\perp}$.



Problem 5. [20 points]

Find a basis of the orthogonal complement of each of the following subspaces.

a) $\mathrm{Nul}\begin{pmatrix} 1 & 0 & 2 & 3 \\ 2 & 1 & 4 & 1 \end{pmatrix}$

b) $\mathrm{Col}\begin{pmatrix} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 3 & 0 & 6 \\ 4 & -1 & 11 \end{pmatrix}$

c) The subspace of all vectors in $\mathbf{R}^4$ whose entries sum to zero.

d) The line $\{(t, 2t, 3t) : t \in \mathbf{R}\}$.

e) $\mathbf{R}^3$

Problem 6. [16 points]

a) Let A be an $m \times n$ matrix of rank r. Which of the following statements are equivalent to "A has full row rank"?

(1) $\text{Nul}(A^T) = \{0\}$
(2) $n = r$
(3) $\text{Col}(A) = \mathbf{R}^m$
(4) A has linearly independent columns
(5) A has a pivot in every row
(6) A is invertible
(7) $Ax = b$ is consistent for every vector b

b) Explain why the projection matrix P_V onto a subspace V can be written as QQ^T for some matrix Q with orthonormal columns. (What is Q in terms of V?)

c) Find three nonzero vectors $v_1, v_2, v_3 \in \mathbf{R}^3$ such that $\{v_1, v_2, v_3\}$ is linearly *de*pendent, but v_3 is not in $\text{Span}\{v_1, v_2\}$. Be sure to label which is v_3.

d) Give an example of a 4×4 matrix A such that $\text{Nul}(A) = \text{Row}(A)$, or explain why no such matrix exists.