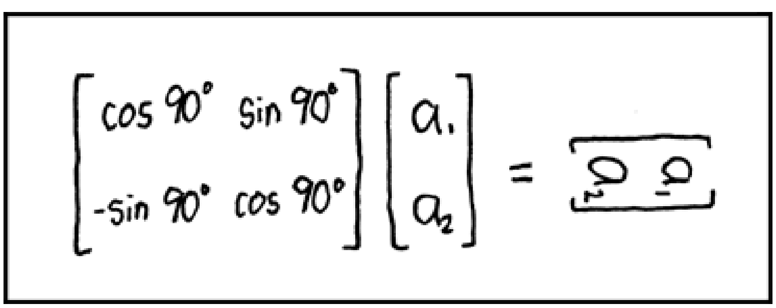
MATH 218D-1
MIDTERM EXAMINATION 2

Name		**Duke NetID**	

Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

Problem 1.

[20 points]

Let L be the line

$$L = \left\{ \begin{pmatrix} t \\ -t \\ 2t \\ 0 \end{pmatrix} : t \in \mathbf{R} \right\}$$

and let $V = L^{\perp}$, the orthogonal complement of L.

a) Find a basis for L.

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix} \right\}$$

b) Find a basis for V.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

c) Find an *orthogonal* basis for V.

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

d) Compute the projection matrix P_V.

$$P_V = \frac{1}{6} \begin{pmatrix} 5 & 1 & -2 & 0 \\ 1 & 5 & 2 & 0 \\ -2 & 2 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Note that it is much easier to compute $P_{V^\perp} = P_L$.

e) Find the orthogonal projection b_V of the vector $b = (1, 1, 3, 1)$ onto V.

$$b_V = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 1 \end{pmatrix}$$

Problem 2. [15 points]

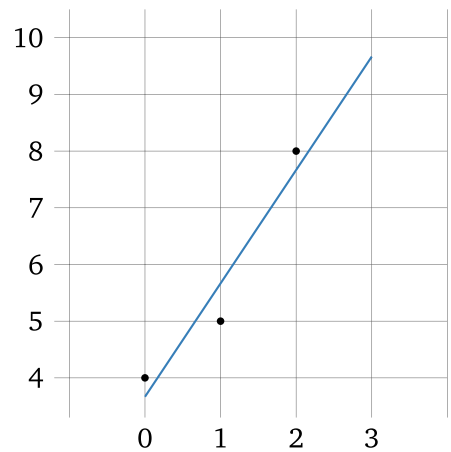
Your blockmate Karxon is currently taking Math 105L. Karxon received a score of 4 on the first exam, 5 on the second, and 8 on the third (out of 10 points total). Not having taken linear algebra yet, Karxon does not know what kind of score to expect on the final exam. Luckily, you can help out.

a) The general equation of a line in $\mathbf{R}^2$ is $y = Cx + D$. Write down the system of linear equations in C and D that would be satisfied by a line passing through the points $(0,4)$, $(1,5)$, and $(2,8)$, and then write down the corresponding matrix equation.

System of equations $\begin{cases} 4 = 0C + D \\ 5 = 1C + D \\ 8 = 2C + D \end{cases} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 8 \end{pmatrix}$

b) Solve the corresponding least squares problem for C and D, and use this to *write down* and *draw* the the best fit line below. (Sorry for the fractions! The final answer isn't too ugly.)

$$y = 2x + 11/3$$



c) This line predicts a score of $9\frac{2}{3}$ points on the fourth (final) exam.

Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) Find the QR decomposition of A.

$$Q = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 0 \\ 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ 0 & 1/\sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} \sqrt{2} & -\sqrt{2} & 0 \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

b) Find the least-squares solution of $Ax = (1,1,3,1)$, using part (a) or otherwise.

$$\widehat{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

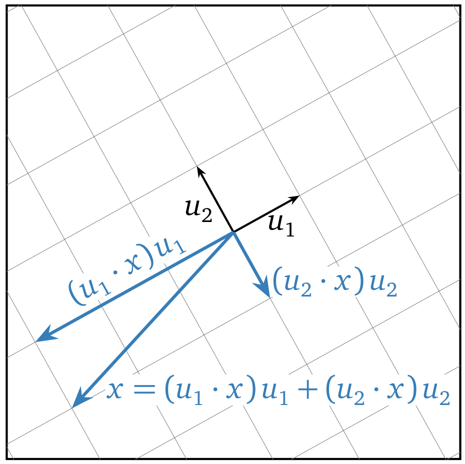
c) Briefly explain how a computer would use your answer to part (a) to compute the answer to part (b) more quickly.

A computer would solve $R\widehat{x} = Q^T b$ using back-substitution, avoiding the use of elimination.

Problem 4.

[15 points]

Two *orthogonal unit* vectors u_1, u_2 in $\mathbf{R}^2$ are drawn in the grid below, along with a vector x.



a) *Draw* and *label* the vectors $(u_1 \cdot x)u_1$ and $(u_2 \cdot x)u_2$.

b) *Draw* and *label* the sum $(u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$.

c) Briefly explain how you can determine $(u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$ without the picture (i.e., only knowing that u_1, u_2 are orthogonal unit vectors in $\mathbf{R}^2$).

Since $\{u_1, u_2\}$ is orthonormal, $(u_1 \cdot x)u_1 + (u_2 \cdot x)u_2$ is equal to the orthogonal projection of x onto $\mathrm{Span}\{u_1, u_2\} = \mathbf{R}^2$, and is thus equal to x.

Problem 5. [20 points]

Short-answer questions: you do not need to justify your answers.

a) Let A be an $n \times n$ matrix that is *not* invertible. Which of the following can you conclude? (Fill in the circles of all that apply.)

- ● A has linearly dependent rows.
- ○ A has two identical columns.
- ○ A has a row of zeros.
- ● The rank of A is less than n.
- ● There are two different vectors x and y in $\mathbf{R}^n$ such that $Ax = Ay$.

b) Let V be a subspace of $\mathbf{R}^n$. In each box below, write $\{0\}$, V, $V^\perp$, or $\mathbf{R}^n$ to make the equality true.

$$\mathrm{Nul}(P_V) = V^\perp \qquad \mathrm{Col}(P_V) = V \qquad \mathrm{Col}(P_V P_{V^\perp}) = \{0\}$$

$$\mathrm{Nul}(P_V^2) = V^\perp \qquad \mathrm{Col}(P_V + P_{V^\perp}) = \mathbf{R}^n.$$

c) Let V be the subspace of $\mathbf{R}^3$ consisting of all vectors (x, y, z) such that $z = 0$ (otherwise known as the xy-plane). Which of the following are bases of V? (Fill in the circles of all that apply.)

○ $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$

● $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}3\\2\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}2\\4\\0\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right\}$

d) If A is a 5×6 matrix of rank 2, then $\mathrm{Col}(A)^\perp$ is a subspace of $\mathbf{R}^5$ of dimension 3.

Problem 6. [15 points]

All of the following statements are false. Find a counterexample for each.

a) Suppose v_1, v_2, and v_3 are distinct vectors in $\mathbf{R}^3$. If $\{v_1, v_2\}$ is linearly independent and $\{v_2, v_3\}$ is linearly independent, then $\{v_1, v_2, v_3\}$ is linearly independent.

Choose any three coplanar vectors such that no two are collinear. For instance,

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

b) All orthogonal symmetric 2×2 matrices are diagonal.

Any matrix of the form $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$ is symmetric and orthogonal.

c) A matrix with orthonormal columns has full row rank.

Any tall matrix with orthonormal columns is a counterexample; for instance,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

d) If $A^T A$ is invertible then A is invertible.

Any tall matrix with full column rank is a counterexample. For instance, any answer to (c) works.

e) A projection matrix is never invertible.

The only counterexample is $P_{\mathbf{R}^n} = I_n$.