MATH 218D-1
PRACTICE MIDTERM EXAMINATION 3

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 75 minutes, without notes or distractions.

Problem 1. [20 points]

Consider the quadratic form

$$q(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 5x_3^2 + 2x_1x_2 - 8x_1x_3 + 8x_2x_3.$$

a) Find a symmetric matrix S such that $q(x) = x^T Sx$.

b) Compute the characteristic polynomial $p(\lambda)$ of S.

The roots of $p(\lambda)$ are $9, 3$, and -3.

c) Find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

d) Find a change of coordinates y_1, y_2, y_3 such that

$$q(x_1, x_2, x_3) = 9y_1^2 + 3y_2^2 - 3y_3^2.$$

(The x_i should be linear functions of the y_i.)

e) What are the minimum and maximum values of $q(x)$ subject to $\|x\| = 1$? For which values of x are those values attained?

Your answers should involve square roots and fractions, *not* decimals.

Solution.

a)
$$S = \begin{pmatrix} 2 & 1 & -4 \\ 1 & 2 & 4 \\ -4 & 4 & 5 \end{pmatrix}$$

b)
$$p(\lambda) = -\lambda^3 + 9\lambda^2 + 9\lambda - 81$$

c)
$$Q = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \qquad D = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

d) $x_1 = -\frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 + \frac{1}{\sqrt{3}}y_3 \quad x_2 = \frac{1}{\sqrt{6}}y_1 + \frac{1}{\sqrt{2}}y_2 - \frac{1}{\sqrt{3}}y_3 \quad x_3 = \frac{2}{\sqrt{6}}y_1 + \frac{1}{\sqrt{3}}y_3$

e) The minimum value is -3, which is attained at $\pm\frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. The maximum value is 9, which is attained at $\pm\frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$.

Problem 2. [10 points]

Consider the symmetric matrix

$$S = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

a) Verify that S is positive-definite without finding its eigenvalues.

b) Compute the LDL^T and Cholesky decompositions of S:

$$S = LDL^T \qquad S = L_1 L_1^T.$$

Solution.

a) This can be accomplished by finding the LU decomposition, which we do in **b)**.

b) We have $S = LDL^T$ for

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

We also have $S = L_1 L_1^T$ for

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 2 & \sqrt{2} & 0 \\ 0 & -\sqrt{2} & \sqrt{3} \end{pmatrix}.$$

Problem 3.

[20 points]

Consider the difference equation

$$\begin{aligned} x_{n+1} &= 2x_n - \phantom{\tfrac{1}{2}}y_n \qquad & x_0 &= 1 \\ y_{n+1} &= \tfrac{3}{2}x_n - \tfrac{1}{2}y_n \qquad & y_0 &= 2. \end{aligned}$$

a) Find a matrix A such that

$$A\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix}.$$

b) Find the eigenvalues of A, and find corresponding eigenvectors.

c) Find a formula for $\binom{x_n}{y_n}$ in terms of n.

d) What is $\lim_{n\to\infty} \binom{x_n}{y_n}$?

e) Solve the following initial value problem:

$$\begin{aligned} u_1'(t) &= 2u_1(t) - \phantom{\tfrac{1}{2}}u_2(t) \qquad & u_1(0) &= 1 \\ u_2'(t) &= \tfrac{3}{2}u_1(t) - \tfrac{1}{2}u_2(t) \qquad & u_2(0) &= 2. \end{aligned}$$

Solution.

a) The matrix is $A = \begin{pmatrix} 2 & -1 \\ 3/2 & -1/2 \end{pmatrix}$.

b) The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1/2$, and corresponding eigenvectors are $w_1 = \binom{1}{1}$ and $w_2 = \binom{2}{3}$.

c) We have $\binom{x_0}{y_0} = -w_1 + w_2$, so

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = -A^n w_1 + A^n w_2 = -w_1 + \frac{1}{2^n} w_2 = -\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2^n}\begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

d) The limit is $-w_1 = -\binom{1}{1}$.

e) The solution is

$$\begin{aligned} u_1(t) &= -e^t + 2e^{t/2} \\ u_2(t) &= -e^t + 3e^{t/2}. \end{aligned}$$

Problem 4. [20 points]

Give examples of matrices with each of the following properties. If no such matrix exists, explain why. *All matrices in this problem have real entries.*

a) A symmetric matrix satisfying

$$S\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix} \quad \text{and} \quad S\begin{pmatrix}2\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\-1\\0\end{pmatrix}.$$

b) A 2×2 matrix whose 1-eigenspace is the line $x+2y=0$ and whose 2-eigenspace is the line $x+3y=0$.

c) A 2×2 matrix that is neither invertible nor diagonalizable.

d) A 2×2 non-invertible matrix with eigenvalue $2+3i$.

e) A 2×2 matrix A that is diagonalizable over $\mathbf{R}$, such that A^2 is not diagonalizable.

Solution.

a) Does not exist: eigenvectors with different eigenvalues would have to be orthogonal.

b) This matrix satisfies

$$A=\begin{pmatrix}-2&-3\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&2\end{pmatrix}\begin{pmatrix}-2&-3\\1&1\end{pmatrix}^{-1}=\begin{pmatrix}4&6\\-1&-1\end{pmatrix}.$$

c) One example is $\begin{pmatrix}0&1\\0&0\end{pmatrix}$.

d) Does not exist: the other eigenvalue would be $2-3i$, so 0 is not an eigenvalue.

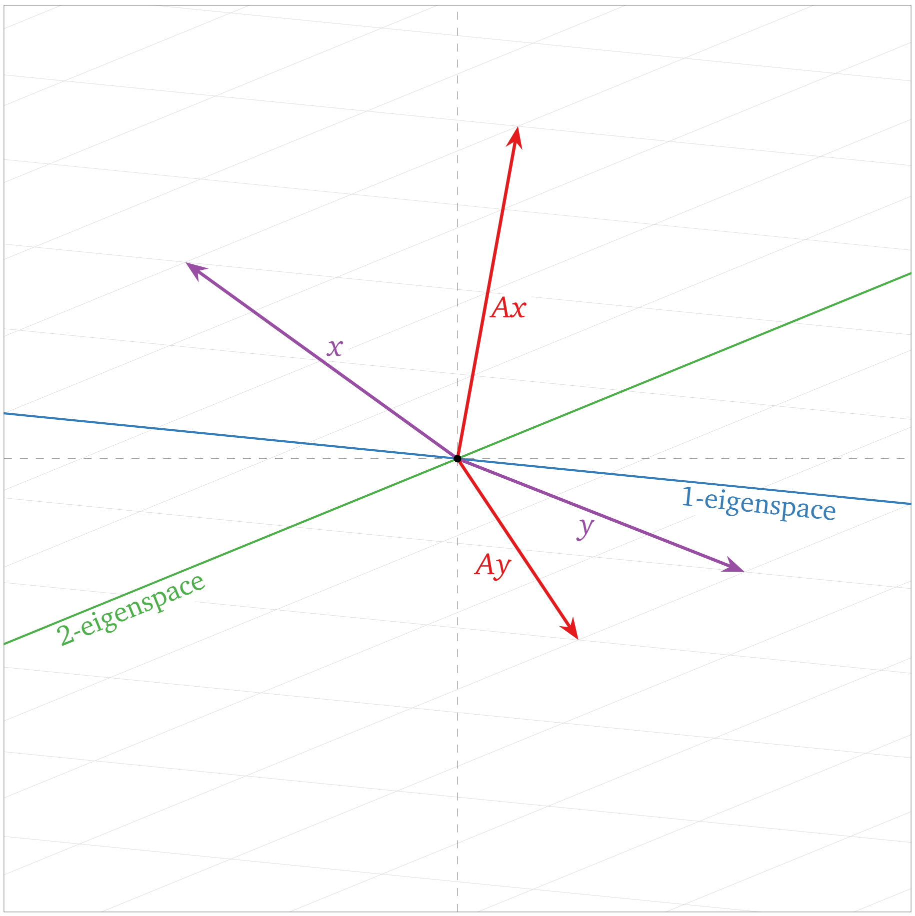
e) Does not exist: if $A=CDC^{-1}$ then $A^2=CD^2C^{-1}$.

Problem 5. [10 points]

A certain diagonalizable 2×2 matrix A has eigenvalues 1 and 2, with eigenspaces drawn below.

a) Draw Ax and Ay on the diagram.

b) For which vectors u is $\|A^n u\|$ bounded? In other words, for which vectors u does $\|A^n u\|$ not approach ∞ as $n \to \infty$?



Solution.

b) Such a u must be a 1-eigenvector.

Problem 6.

[20 points]

In this problem, you need not explain your answers; just write them in the spaces provided.

Let A be an $n \times n$ matrix with real entries.

a) Which **one** of the following statements is correct? | 2 |

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct? | 2 |

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

c) Which of the following 3×3 matrices are necessarily diagonalizable over the real numbers? (List all that apply.) | 1, 2, 3 |

1. A matrix with three distinct real eigenvalues.
2. A symmetric matrix with two real eigenvalues.
3. A matrix with a real eigenvalue λ of algebraic multiplicity 2, such that the λ-eigenspace has dimension 2.
4. A matrix with a real eigenvalue λ such that the λ-eigenspace has dimension 2.

d) Suppose that the characteristic polynomial of A is

$$p(\lambda) = \lambda(\lambda - 2)(\lambda - 3)^2.$$

Which of the following can you determine from this information? (Circle all that apply.)

(1) The number n.
(2) The trace of A.
(3) The determinant of A.
(4) The rank of A.
(5) Whether A is symmetric.
(6) Whether A is diagonalizable.
(7) The eigenvalues of A.