

Homework #10

due Wednesday, March 23, at 11:59pm

1. For each matrix A and each vector v , decide if v is an eigenvector of A , and if so, find the eigenvalue λ .

a) $\begin{pmatrix} -20 & 42 & 58 \\ 1 & -1 & -3 \\ -1 & 18 & 26 \end{pmatrix}, \begin{pmatrix} 1 \\ 5 \\ -2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 3 & 0 \\ -5 & 4 & 2 \\ 3 & 3 & 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c) $\begin{pmatrix} -7 & 32 & -76 \\ 7 & -22 & 59 \\ 3 & -11 & 28 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

e) $\begin{pmatrix} -3 & 2 & -3 \\ 3 & -3 & -2 \\ -4 & 2 & -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

2. Suppose that A is an $n \times n$ matrix such that $Av = 2v$ for some $v \neq 0$. Let C be any invertible matrix. Consider the matrices

a) A^{-1} b) $A + 2I_n$ c) A^3 d) CAC^{-1} .

Show that v is an eigenvector of **a)–c)** and that Cv is an eigenvector of **d)**, and find the eigenvalues.

3. Here is a handy trick for computing eigenvectors of a 2×2 matrix.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix with eigenvalue λ . Explain why $\begin{pmatrix} -b \\ a-\lambda \end{pmatrix}$ and $\begin{pmatrix} d-\lambda \\ -c \end{pmatrix}$ are λ -eigenvectors of A if they are nonzero.

For which matrices A does this trick fail?

4. For each 2×2 matrix A , **i)** compute the characteristic polynomial using the formula $p(\lambda) = \lambda^2 - \text{Tr}(A)\lambda + \det(A)$. Use this to **ii)** find all real eigenvalues, and **iii)** find a basis for each eigenspace, using Problem 3 when applicable. **iv)** Draw and label each eigenspace. **v)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

a) $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ b) $\begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}$ c) $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ d) $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ e) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

5. For each matrix A , **i)** find all real eigenvalues of A , and **ii)** find a basis for each eigenspace. **iii)** Is the matrix diagonalizable (over the real numbers)? If so, find an invertible matrix C and a diagonal matrix D such that $A = CDC^{-1}$.

At least one eigenvalue is provided; use that and **synthetic division** (or a computer algebra system) to find the others.

$$\text{a) } \begin{pmatrix} -1 & 7 & 5 \\ 0 & 1 & -2 \\ 0 & 1 & 4 \end{pmatrix}, \lambda = -1 \quad \text{b) } \begin{pmatrix} 7 & 12 & 12 \\ -8 & -13 & -12 \\ 4 & 6 & 5 \end{pmatrix}, \lambda = 1$$

$$\text{c) } \begin{pmatrix} 6 & 2 & 3 \\ -14 & -7 & -12 \\ 1 & 2 & 4 \end{pmatrix}, \lambda = 1$$

Optional (if you want more practice):

$$\text{d) } \begin{pmatrix} -11 & -54 & 10 \\ -2 & -7 & 2 \\ -21 & -90 & 20 \end{pmatrix}, \lambda = 1 \quad \text{e) } \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \lambda = 2$$

$$\text{f) } \begin{pmatrix} 13 & 18 & -18 \\ -12 & -17 & 18 \\ -4 & -6 & 7 \end{pmatrix}, \lambda = 1 \quad \text{g) } \begin{pmatrix} -10 & 28 & -18 & -76 \\ -1 & 9 & -6 & -2 \\ 4 & -8 & 7 & 26 \\ 0 & 2 & -2 & 4 \end{pmatrix}, \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix}$$

6. Consider the matrix

$$A = \begin{pmatrix} 4 & -3 & 0 \\ 2 & -1 & 0 \\ 1 & -1 & 1 \end{pmatrix}.$$

- a) Find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.
 b) Find a *different* diagonal matrix D' and a *different* invertible matrix C' such that $A = C'D'C'^{-1}$.

[**Hint:** Try re-ordering the eigenvalues.]

7. Compute the matrix with eigenvalues 0, 1, 2 and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}.$$

(There is only one such matrix.)

8. a) Show that A and A^T have the same eigenvalues.
 b) Give an example of a 2×2 matrix A such that A and A^T do not share any eigenvectors.
 c) A *stochastic matrix* is a matrix with nonnegative entries whose columns sum to 1. Explain why 1 is an eigenvalue of a stochastic matrix.

[**Hint:** show that $(1, 1, \dots, 1)$ is an eigenvector of A^T .]

9. a) Find all eigenvalues of the matrix

$$\begin{pmatrix} 1 & -1 & 2 & 3 & 4 \\ 0 & 3 & -1 & -2 & -5 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

- b) Explain how to find the eigenvalues of any triangular matrix.

10. Let A be an $n \times n$ matrix, and let C be an invertible $n \times n$ matrix. Prove that the characteristic polynomial of CAC^{-1} equals the characteristic polynomial of A .

In particular, A and CAC^{-1} have the same eigenvalues, the same determinant, and the same trace. They are called *similar* matrices.

11. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns. Prove that any real eigenvalue of an orthogonal matrix Q is ± 1 .

12. Let V be the plane $x + y + z = 0$, and let $R_V = I_3 - 2P_{V^\perp}$ be the reflection matrix over V , as in HW9#11. Diagonalize R_V without doing any computations.

13. The *Fibonacci numbers* are defined recursively as follows:

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geq 0).$$

The first few Fibonacci numbers are $0, 1, 1, 2, 3, 5, 8, 13, \dots$. In this problem, you will find a closed formula (as opposed to a recursive formula) for the n th Fibonacci number by solving a difference equation.

- a) Let $v_n = \begin{pmatrix} F_{n+1} \\ F_n \end{pmatrix}$, so $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, etc. Find a state change matrix A such that $v_{n+1} = Av_n$ for all $n \geq 0$.

- b) Show that the eigenvalues of A are $\lambda_1 = \frac{1}{2}(1 + \sqrt{5})$ and $\lambda_2 = \frac{1}{2}(1 - \sqrt{5})$, with corresponding eigenvectors $w_1 = \begin{pmatrix} -1 \\ \lambda_2 \end{pmatrix}$ and $w_2 = \begin{pmatrix} -1 \\ \lambda_1 \end{pmatrix}$.

[Hint: Check that $Aw_i = \lambda_i w_i$ using the relations $\lambda_1 \lambda_2 = -1$ and $\lambda_1 + \lambda_2 = 1$.]

- c) Expand v_0 in this eigenbasis: that is, find x_1, x_2 such that $v_0 = x_1 w_1 + x_2 w_2$. (It helps to write x_1, x_2 in terms of λ_1, λ_2 .)

- d) Multiply $v_0 = x_1 w_1 + x_2 w_2$ by A^n to show that

$$F_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}.$$

- e) Use this formula to explain why F_{n+1}/F_n approaches the **golden ratio** when n is large.

14. Pretend that there are three **Red Box** kiosks in Durham. Let x_t, y_t, z_t be the number of copies of **Prognosis Negative** at each of the three kiosks, respectively, on day t . Suppose in addition that a customer renting a movie from kiosk i will return the movie the next day to kiosk j , with the following probabilities:

		Renting from kiosk		
		1	2	3
Returning to kiosk	1	30%	40%	50%
	2	30%	40%	30%
	3	40%	20%	20%

For instance, a customer renting from kiosk 3 has a 50% probability of returning it to kiosk 1.

- a) Let $v_t = (x_t, y_t, z_t)$. Find the state change matrix A such that $v_{t+1} = Av_t$.
- b) Diagonalize A . What are its eigenvalues?
 [Hint: A is a stochastic matrix, so you know one eigenvalue by Problem 8(c).]
- c) If you start with a total of 1 000 copies of Prognosis Negative, how many of them will eventually end up at each kiosk? Does it matter what the initial state is?

This is an example of a **stochastic process**, and is an important application of eigenvalues and eigenvectors.

15. Let $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Find a closed formula for A^n : that is, an expression of the form

$$A^n = \begin{pmatrix} a_{11}(n) & a_{12}(n) \\ a_{21}(n) & a_{22}(n) \end{pmatrix},$$

where $a_{ij}(n)$ is a function of n .

16. Give an example of each of the following, or explain why no such example exists.
- a) An invertible matrix with characteristic polynomial $p(\lambda) = -\lambda^3 + 2\lambda^2 + 3\lambda$.
 - b) A 2×2 orthogonal matrix with no real eigenvalues.
17. Suppose that A is a square matrix such that A^k is the zero matrix for some $k > 0$. Show that 0 is the only eigenvalue of A .

- 18.** Decide if each statement is true or false, and explain why.
- a) If v, w are eigenvectors of a matrix A , then so is $v + w$.
 - b) An eigenvalue of $A + B$ is the sum of an eigenvalue of A and an eigenvalue of B .
 - c) An eigenvalue of AB is the product of an eigenvalue of A and an eigenvalue of B .
 - d) If $Ax = \lambda x$ for some vector x , then λ is an eigenvalue of A .
 - e) A matrix with eigenvalue 0 is not invertible.
 - f) The eigenvalues of A are equal to the eigenvalues of a row echelon form of A .
 - g) If v, w are eigenvectors of A with different eigenvalues, then $\{v, w\}$ is linearly independent.