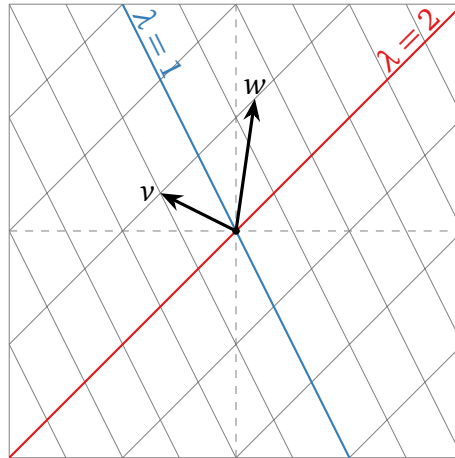


## Homework #11

due Wednesday, March 30, at 11:59pm

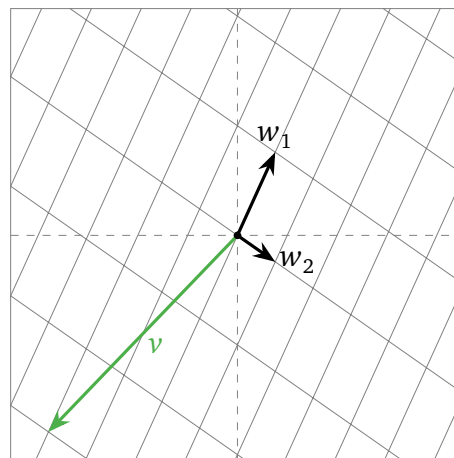
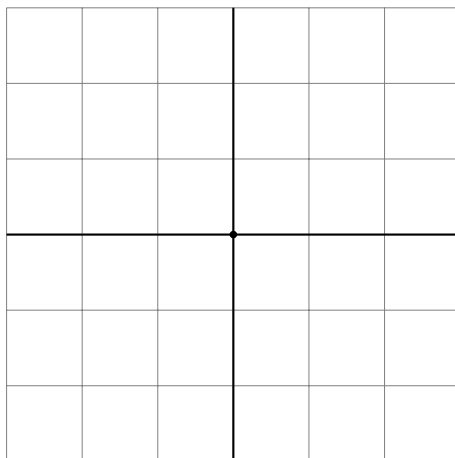
1. A certain  $2 \times 2$  matrix  $A$  has eigenvalues 1 and 2. The eigenspaces are shown in the picture below.

- a) Draw  $Av$ ,  $A^2v$ , and  $Aw$ .
- b) Compute the limit of  $A^n v / \|A^n v\|$  as  $n \rightarrow \infty$ .



2. A certain diagonalizable  $2 \times 2$  matrix  $A$  is equal to  $CDC^{-1}$ , where  $C$  has columns  $w_1, w_2$  pictured below, and  $D = \begin{pmatrix} 1/3 & 0 \\ 0 & 1/2 \end{pmatrix}$ .

- a) Draw  $C^{-1}v$  on the left.
- b) Draw  $DC^{-1}v$  on the left.
- c) Draw  $Av = CDC^{-1}v$  on the right.
- d) What happens to  $A^n v$  as  $n \rightarrow \infty$ ?



3. Compute the following complex numbers.

$$\begin{array}{llll} \text{a)} (1+i) + (2-i) & \text{b)} (1+i)(2-i) & \text{c)} \overline{2-i} & \text{d)} \frac{1+i}{2-i} \\ \text{e)} |1+i| & \text{f)} 2e^{2\pi i/3} & \text{g)} 5e^{3\pi i} & \end{array}$$

4. Express each complex number in polar coordinates  $re^{i\theta}$ .

$$\text{a)} 1+i \quad \text{b)} \frac{-1+i\sqrt{3}}{2} \quad \text{c)} -\sqrt{3}-3i \quad \text{d)} \frac{1}{1+i} \quad \text{e)} (1-i\sqrt{3})^n$$

5. For which numbers  $\theta$  is  $e^{i\theta} = 1$ ? What about  $-1$ ?

6. For each matrix  $A$  and each vector  $x$ , decide if  $x$  is an eigenvector of  $A$ , and if so, find the eigenvalue  $\lambda$ .

$$\begin{array}{ll} \text{a)} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}, \begin{pmatrix} i \\ 1 \end{pmatrix} & \text{b)} \begin{pmatrix} -4 & 13 & 13 \\ 2 & -2 & -4 \\ -4 & 8 & 10 \end{pmatrix}, \begin{pmatrix} 1+5i \\ -2i \\ 4i \end{pmatrix} \\ \text{c)} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -3 & -3 \\ -2 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 2+i \\ 1 \\ -i \end{pmatrix} & \end{array}$$

Careful! It is difficult to recognize by inspection if two complex vectors are (complex) scalar multiples of each other.

7. For each  $2 \times 2$  matrix  $A$ , **i)** compute the characteristic polynomial, **ii)** find all (real and complex) eigenvalues, and **iii)** find a basis for each eigenspace, using HW10#3 when applicable. **iv)** Is the matrix diagonalizable (over the complex numbers)? If so, find an invertible matrix  $C$  and a diagonal matrix  $D$  such that  $A = CDC^{-1}$ .

$$\text{a)} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \text{b)} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \quad \text{c)} \begin{pmatrix} -3 & 5 \\ -10 & 7 \end{pmatrix}$$

8. Diagonalize the following matrix over the complex numbers:

$$A = \begin{pmatrix} 1 & 4 & -6 \\ -6 & 7 & -22 \\ -2 & 1 & -5 \end{pmatrix}.$$

One eigenvalue is  $\lambda = -1$ .<sup>1</sup>

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<sup>1</sup>This problem is included to make you do Gaussian elimination with complex numbers *one time*, so that you'll be grateful to have computers do it for you in the future.

9. A certain forest contains a population of rabbits and a population of foxes. If there are  $r_n$  rabbits and  $f_n$  foxes in year  $n$ , then

$$\begin{aligned}r_{n+1} &= 3r_n - f_n \\ f_{n+1} &= r_n + 2f_n\end{aligned}$$

in other words, each rabbit produces three baby rabbits on average, but there is some loss due to predation by foxes; each fox produces two babies on average, but this is increased with ample prey.

- Let  $v_n = \begin{pmatrix} r_n \\ f_n \end{pmatrix}$ . Find a matrix  $A$  such that  $v_{n+1} = Av_n$ .
- Find an eigenbasis of  $A$ . (The eigenvectors and eigenvalues will be complex.)  
[Hint: Part d) will be easier if you choose the eigenvectors with first coordinate equal to 1.]
- Suppose that  $r_0 = 2$  and  $f_0 = 1$ . Find closed formulas for  $r_n$  and  $f_n$ . Find a formula for  $r_n$  involving only real numbers. (This latter formula can involve an arctan.)
- In this model, the populations do not stabilize. How many years will it take for the foxes to eat all of the rabbits?

In general, any  $2 \times 2$  difference equation with a complex eigenvalue will exhibit oscillation centered at zero. This phenomenon can be described explicitly, but is beyond the scope of this course.

10.
  - Let  $A$  be an  $n \times n$  matrix. Prove that  $\lambda$  is an eigenvalue of  $A$  with geometric multiplicity  $n$  if and only if  $A = \lambda I_n$ .
  - Find a non-diagonal  $2 \times 2$  matrix such that 1 is an eigenvalue with algebraic multiplicity 2.
11. Find examples of real  $2 \times 2$  matrices  $A$  with the following properties.
- $A$  is invertible and diagonalizable over the real numbers.
  - $A$  is invertible but not diagonalizable over the complex numbers.
  - $A$  is diagonalizable over the real numbers but not invertible.
  - $A$  is neither invertible nor diagonalizable over the complex numbers.

This shows that *invertibility and diagonalizability have nothing to do with each other*.

12. Let  $A$  be an  $n \times n$  matrix.
- Show that the product of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to  $\det(A)$ .
  - [Optional] Show that the sum of the (real and complex) eigenvalues, counted with algebraic multiplicity, is equal to  $\text{Tr}(A)$ .

(Both of these are identities involving the characteristic polynomial of  $A$ .)

13. For each matrix in HW10#5(a)–(c), compute the algebraic and geometric multiplicity of each eigenvalue. What does your answer say about diagonalizability?

**Optional:** do (d)–(g) as well.

14. Give an example of each of the following, or explain why no such example exists. All matrices should have real entries.

- a) A  $3 \times 3$  matrix with eigenvalues 0, 1, 2, and corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

- b) A  $4 \times 4$  matrix having eigenvalue 2 with algebraic multiplicity 2 and geometric multiplicity 3.
- c) A  $3 \times 3$  matrix with one complex (non-real) eigenvalue and two real eigenvalues.
- d) A  $2 \times 2$  matrix  $A$  such that  $A^2$  is diagonalizable over the real numbers but  $A$  is not diagonalizable, even over the complex numbers.  
[Hint: try a nonzero matrix  $A$  such that  $A^2 = 0$ .]

15. Decide if each statement is true or false, and explain why.

- a) If  $A$  and  $B$  are diagonalizable  $n \times n$  matrices, then so is  $AB$ .
- b) An  $n \times n$  matrix with  $n$  (different) eigenvalues is diagonalizable.
- c) An  $n \times n$  matrix is diagonalizable if it has  $n$  eigenvalues, counted with algebraic multiplicity.
- d) Any  $2 \times 2$  real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
- e) Any  $3 \times 3$  real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
- f) Any  $4 \times 4$  real matrix with a complex (non-real) eigenvalue is diagonalizable over the complex numbers.
- g) Any  $2 \times 2$  real matrix has a real eigenvalue.
- h) Any  $3 \times 3$  real matrix has a real eigenvalue.
- i) Any  $n \times n$  matrix has a (real or complex) eigenvalue.
- j) If the characteristic polynomial of  $A$  is  $-(\lambda^3 - 1) = -(\lambda^2 + \lambda + 1)(\lambda - 1)$ , then the 1-eigenspace of  $A$  is a line.

**16.** Solve the following initial value problems. Your solutions should involve only real numbers.

$$\text{a) } \begin{cases} u_1' = u_1 - 2u_2 & u_1(0) = -3 \\ u_2' = u_1 + 4u_2 & u_2(0) = 2 \end{cases} \quad \text{b) } \begin{cases} u_1' = 3u_1 - u_2 & u_1(0) = 4 \\ u_2' = u_1 + 2u_2 & u_2(0) = 2 \end{cases}$$

**17.** Solve the following initial value problem.

$$p''(t) = -2p(t) + 3p'(t) \quad p(0) = 1 \quad p'(0) = -1.$$

**18.** Use the matrix exponential to solve the following initial value problem.

$$\begin{cases} u_1' = 2u_2 + u_3 & u_1(0) = 2 \\ u_2' = -u_3 & u_2(0) = 3 \\ u_3' = 0 & u_3(0) = -1. \end{cases}$$

(This is one of the few instances where the matrix exponential leads to a computable solution!)