Homework #12

due Wednesday, April 6, at 11:59pm

1. For each symmetric matrix S, find an orthogonal matrix Q and a diagonal matrix D such that $S = QDQ^T$.

$$\textbf{a)}\ \begin{pmatrix} 1 & -3 \\ -3 & 1 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 1 & -8 & 4 \\ -8 & 1 & 4 \\ 4 & 4 & 7 \end{pmatrix}$$

The eigenvalues in **d)** are $3, 6, 9$ and in **e)** are $-9, 9$.

2. For each matrix S of Problem 1, decide if S is positive-semidefinite, and if so, compute its positive-semidefinite square root $\sqrt{S} = Q\sqrt{D}Q^T$. Verify that $(\sqrt{S})^2 = S$.

Remark: Since $\sqrt{S}$ is also symmetric, we have $S = \sqrt{S}^T\sqrt{S}$, so this is another way to factorize a positive-semidefinite matrix as A^TA.

3. Consider the matrix

$$S = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

of Problem 1(d). Write S in the form $\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \lambda_3 u_3 u_3^T$ for numbers $\lambda_1, \lambda_2, \lambda_3$ and orthonormal vectors u_1, u_2, u_3.

4. Find *all possible* orthogonal diagonalizations

$$\frac{1}{5}\begin{pmatrix} 41 & 12 \\ 12 & 34 \end{pmatrix} = QDQ^T.$$

5. Let S be a symmetric matrix such that $S^k = 0$ for some $k > 0$. Show that $S = 0$.
[**Hint:** Use HW10#17.]

6. Let S be a symmetric orthogonal 2×2 matrix.

a) Show that $S = \pm I_2$ if it has only one eigenvalue.
[**Hint:** See HW10#11.]

b) Suppose that S has two eigenvalues. Show that S is the matrix for the reflection over a line L in $\mathbf{R}^2$. (Recall that the reflection over a line L is given by $R_L = I_2 - 2P_{L^\perp}$.)
[**Hint:** Write S as $\lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T$, and use the projection formula to write I_2 and $P_{L^\perp}$ in this form as well. What is L?]

7. **a)** Let S be a diagonalizable (over $\mathbf{R}$) $n \times n$ matrix with orthogonal eigenspaces: that is, eigenspaces with different eigenvalues are orthogonal subspaces. Prove that S is symmetric.

[**Hint:** choose *orthonormal* bases for each eigenspace.]

b) Let S be a matrix that can be written in the form

$$S = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots + \lambda_n q_n q_n^T$$

for some vectors $q_1, q_2, \ldots, q_n$. Prove that S is symmetric.

c) Let V be a subspace of $\mathbf{R}^n$, and let P_V be the projection matrix onto V. Use **a)** or **b)** to prove that P_V is symmetric. (We proved this in class using the formula $P_V = A(A^T A)^{-1} A^T$.)

8. For which matrices A is $S = A^T A$ positive-definite? If S is not positive-definite, find a vector x such that $x^T S x = 0$. In any case, do not compute S!

$$\textbf{a)} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 0 & 3 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 3 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$$

9. **a)** If S is positive-definite and C is invertible, show that CSC^T is positive-definite.

b) If S and T are positive-definite, show that $S + T$ is positive-definite.

c) If S is positive-definite, show that S is invertible and that S^{-1} is positive-definite.

[**Hint:** For **a)** and **b)** use the positive-energy characterization of positive-definiteness; for **c)** use the positive-eigenvalue characterization.]

10. Let S be a positive-definite matrix.

a) Show that the diagonal entries of S are positive.

[**Hint:** compute $e_i^T S e_i$.]

b) Show that the diagonal entries of S are all greater than or equal to the smallest eigenvalue of S.

[**Hint:** if not, apply **a)** to $S - aI_n$ for a diagonal entry a that is smaller than all eigenvalues.]

11. Decide if each statement is true or false, and explain why. All matrices are real.

a) A symmetric matrix is diagonalizable.

b) If A is any matrix then $A^T A$ is positive-semidefinite.

c) A symmetric matrix with positive determinant is positive-definite.

d) If $A = CDC^{-1}$ for a diagonal matrix D and a non-orthogonal invertible matrix C, then A is not symmetric.

e) A positive-definite matrix has the form $A^T A$ for a matrix A with full column rank.

f) The only positive-definite projection matrix is the identity.

g) All eigenvalues of a positive-definite symmetric matrix are positive real numbers.

12. For each symmetric matrix S, decide if S is positive-definite. If so, find its LDL^T and Cholesky decompositions. Do not compute any eigenvalues!

$$\textbf{a)}\ \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 2 & 0 \\ 2 & 5 & -1 \\ 0 & -1 & 3 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} 3 & -2 & 2 \\ -2 & 4 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 1 & 1 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 2 & 6 & 14 & 8 \\ 1 & 3 & 8 & 9 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} -1 & 2 & 3 & -2 \\ 2 & -3 & -8 & 4 \\ 3 & -8 & -4 & 6 \\ -2 & 4 & 6 & -1 \end{pmatrix}$$

13. Consider the matrix

$$S = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Without multiplying the matrices, find:

a) The determinant of S.

b) The eigenvalues of S.

c) The eigenvectors of S.

d) A reason why S is symmetric positive-definite.

14. **a)** For each symmetric matrix S, compute the associated quadratic form $q(x) = x^T S x$.

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

b) Let A be a square matrix and let $S = \frac{1}{2}(A + A^T)$. Show that S is symmetric and that $x^T A x = x^T S x$. (This is why we only consider symmetric matrices when studying quadratic forms.)

15. For each quadratic form $q(x_1, x_2)$, **i)** write $q(x)$ in the form $x^T S x$ for a symmetric matrix S, **ii)** find coordinates y_1, y_2 such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2$, and **iii)** find the maximum and minimum values of $q(x_1, x_2)$ subject to the constraint $x_1^2 + x_2^2 = 1$, and at which points (x_1, x_2) these values are attained.

a) $q(x_1, x_2) = 14x_1^2 + 4x_1x_2 + 11x_2^2$

b) $q(x_1, x_2) = \dfrac{1}{10}(21x_1^2 - 6x_1x_2 + 29x_2^2)$

c) $q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2$

16. For the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3,$$

find coordinates y_1, y_2, y_3 such that $q(x) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2$, and find the maximum and minimum values of $q(x_1, x_2, x_3)$ subject to the constraint $x_1^2 + x_2^2 + x_3^2 = 1$, along with the points (x_1, x_2, x_3) at which these values are attained.

17. Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3.$$

Find all vectors $x = (x_1, x_2, x_3)$ maximizing $q(x)$ subject to $\|x\| = 1$. (There are infinitely many!)