**Homework #13**

due Friday, April 15, at 11:59pm

**1.** For each quadratic form $q(x_1, x_2)$ of HW12#15**(a,b)**, first **i)** draw the solutions of $q(x_1, x_2) = 1$, being sure to draw the shortest and longest solutions, and then **ii)** find the maximum and minimum values of $\|x\|^2$ subject to the constraint $q(x) = 1$, and at which points $(x_1, x_2)$ these values are attained.

What happens if you try to extremize $\|x\|^2$ subject to

$$q(x_1, x_2) = x_1^2 - 6x_1x_2 + x_2^2 = 1?$$

(This is the form from part **(c)** of HW12#15.)

**2.** For the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3$$

of HW12#16, find the maximum and minimum values of $\|x\|^2$ subject to the constraint $q(x) = 1$, along with the points $(x_1, x_2, x_3)$ at which these values are attained.

**3.** **a)** Consider the quadratic form

$$q(x_1, x_2, x_3) = 7x_1^2 + 6x_2^2 + 5x_3^2 + 4x_1x_2 + 4x_2x_3,$$

of HW12#16. Find the *smallest* value of $q(x)$ subject to the constraints $\|x\| = 1$ *and* $x \perp \frac{1}{3}(1, -2, 2)$. At which vectors $x$ is this minimum attained?

**b)** Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + x_2^2 + 7x_3^2 - 16x_1x_2 + 8x_1x_3 + 8x_2x_3.$$

of HW12#17. Find the *largest* value of $q(x)$ subject to the constraints $\|x\| = 1$ *and* $x \perp \frac{1}{\sqrt{5}}(0, 1, 2)$. At which vectors $x$ is this maximum attained?

**4.** For each matrix $A$, find the minimum and maximum values of $\|Ax\|^2$ subject to the constraint $\|x\| = 1$. At which vectors are these extrema achieved? Check your work by choosing a vector $x$ maximizing $\|Ax\|^2$, computing $b = Ax$, and verifying that $\|b\|^2$ is equal to the maximum.

**a)** $\begin{pmatrix} 3 & -1 \\ 2 & 3 \\ 1 & -1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 0 & -1 \\ 1 & 1 & 2 \\ 1 & -1 & 0 \\ 1 & 2 & 0 \end{pmatrix}$

**5.** Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 & 4 & -3 \\ 1 & 7 & -2 & 3 & -5 \\ 2 & 0 & 8 & -1 & 1 \\ 1 & 2 & 0 & 3 & 9 \end{pmatrix}.$$

**a)** Find a unit vector $u_1$ maximizing $\|Ax\|^2$ subject to $\|x\| = 1$.

**b)** Find the maximum value of $\|Ax\|^2$ subject to $\|x\| = 1$ and $x \perp u_1$.

**c)** Find the minimum value of $\|Ax\|$ subject to $\|x\| = 1$ without doing any work.

You'll need to use a computer algebra system. With the Sage cell on the course webpage, you'd want something like this:

```
A = Matrix([[ 3., 2.,-1., 4.,-3.],
            [ 1., 7.,-2., 3.,-5.],
            [ 2., 0., 8.,-1., 1.],
            [ 1., 2., 0., 3., 9.]])
pprint((A.transpose()*A).eigenvects())
```

(Entering numbers as "`3.`" instead of "3" forces SymPy to perform a floating-point computation instead of a symbolic one.)

**6.** Show that the maximum value of $\|Ax\|$ subject to $\|x\| = 1$ is the same as the maximum value of $\|Ax\|/\|x\|$ subject to $x \neq 0$.

**Remark:** This gives an equivalent definition of the *matrix norm* $\|A\|$.

**7.** In this problem, we will touch on the role of quadratic optimization in *spectral graph theory*. Spectral graph theory is the study of graphs using linear algebra, and is widely applied to problems in networking and partitioning. (Google's PageRank algorithm can be formulated as a spectral graph theory problem.)

A *graph* is a set of *vertices*, or points, connected by a set of *edges*. For simplicity, we will assume that each edge has distinct endpoints (i.e., there are no loop edges), and that there is at most one edge connecting any two vertices: such a graph is called *simple*. Under these assumptions, an edge is determined by the two vertices it connects, so we can write $e = (1, 2)$ for the edge connecting vertices 1 and 2. We also write $i \sim j$ if $(i, j)$ is an edge of $G$. The *degree* of a vertex is the number of edges connected to it; the degree of vertex $i$ is written $\deg(i)$.

Let $G$ be a graph with $n$ vertices labeled $1, 2, \ldots, n$. We consider a vector $x \in \mathbf{R}^n$ as a way to assign a real number to each vertex: the $i$th coordinate $x_i$ is the number attached to the $i$th vertex. The *Laplacian* of $G$ is the $n \times n$ matrix $L$ whose $(i, j)$ entry is

$$L_{ij} = \begin{cases} \deg(i) & \text{if } i = j \\ -1 & \text{if there is an edge from vertex } i \text{ to vertex } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that $L$ is symmetric. Let $x \in \mathbf{R}^n$ and let $y = Lx$. Then the $i$th coordinate of $y$ is

$$(\star) \qquad y_i = x_i \deg(i) - \sum_{j\sim i} x_j = \sum_{j\sim i}(x_i - x_j).$$

In other words, $y$ is the vector that assigns the number $\sum_{j\sim i}(x_i - x_j)$ to vertex $i$.

The eigenvalues of the graph Laplacian contain important information about the structure of the graph.

**a)** Show that the vector $\mathbf{1} = (1, 1, \ldots, 1) \in \mathbf{R}^n$ is in the null space of $L$.

It follows that 0 is always an eigenvalue of $L$.

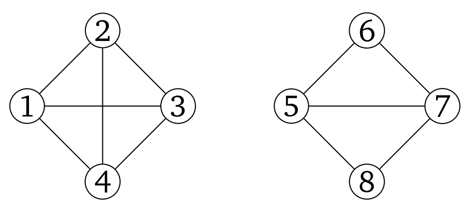
**b)** Show that $x^T Lx = \sum_{j\sim i}(x_i - x_j)^2$. Explain why $L$ is positive-semidefinite.

Since $L$ is positive-semidefinite, all of its eigenvalues are *nonnegative*, so 0 is the smallest eigenvalue of $L$. The fact that 0 is an eigenvalue gives us no information about the graph, so we wish to "rule it out" by imposing the constraint $x \perp \mathbf{1}$.

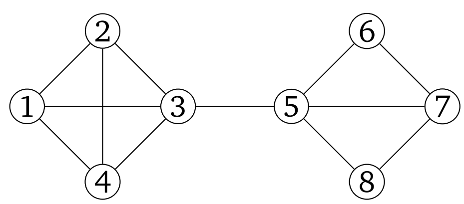
According to **b)**, minimizing $q(x) = x^T Lx$ subject to the constraints $\|x\|^2 = 1$ and $x \perp \mathbf{1}$ amounts to finding a way to assign a number to each vertex such that *neighboring vertices have similar values*, but such that the sum of the values is zero ($x \perp \mathbf{1}$) and the sum of their squares is 1 ($\|x\| = 1$).

For each of the following graphs, **i)** compute the Laplacian matrix $L$ and **ii)** minimize $x^T Lx$ subject to $x \perp \mathbf{1}$ and $\|x\| = 1$. **iii)** For a (unit) vector $x$ achieving this minimum, draw the number $x_i$ next to vertex $i$ on the graph. **iv)** What does the second-smallest eigenvalue say about the graph? (This is open-ended.)

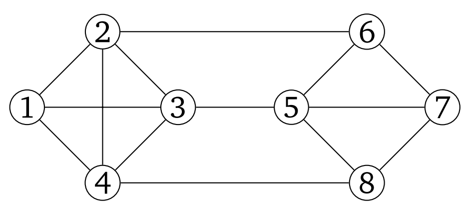
**c)**




**d)**




**e)**




You should feel free to use a computer algebra system to compute the eigenvalues and eigenvectors. For instance, you can use SymPy in the Sage cell on the course webpage. Finding the eigenvalues and eigenvectors of a matrix in SymPy is done as follows: if your matrix is

$$A = \begin{pmatrix} 7 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

then you would type:

```
A = Matrix([[7.,2.,0.],[2.,6.,2.],[0.,2.,5.]])
pprint(A.eigenvects())
```

(Entering numbers as “`3.`” instead of “3” forces SymPy to perform a floating-point computation instead of a symbolic one.) The output is a list of tuples of the form (eigenvalue, multiplicity, eigenspace basis)—note that the eigenspace basis will not necessarily be orthonormal.

**8.** For each matrix $A$, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

**a)** $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$

**d)** $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

[**Hint:** one of the singular values in **e)** is 12.]

**9.** Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 8(a). Let $\sigma_1, \sigma_2$ be the singular values of $A$. Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

**10.** Find the matrix $A$ satisfying

$$A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix},$$

and write the SVD of $A$ in outer product form.

[**Hint:** Start by finding the SVD.]

**11.** Let $A$ be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of $A$ in outer product form.

**12.** Let $S$ be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal eigenbasis, where $v_i$ has eigenvalue $\lambda_i$.

**a)** Show that the singular values of $S$ are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\text{rank}(S) = r$.

**b)** Find the singular value decomposition of $S$ in outer product form, in terms of the $\lambda_i$ and the $v_i$.

**13.** **a)** Show that all singular values of an orthogonal matrix are equal to 1.

**b)** Let $A$ be an $m \times n$ matrix, let $Q_1$ be an $m \times m$ orthogonal matrix, and let $Q_2$ be an $n \times n$ orthogonal matrix. Show that $A$ has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use HW10#10.]

**Remark:** This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

**14.** Let $A$ be a matrix of full column rank and let $A = QR$ be the $QR$ decomposition of $A$.

**a)** Show that $A$ and $R$ have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

**b)** What is the relationship between the left singular vectors of $A$ and $R$?

**15.** Let $A$ be a matrix with first singular value $\sigma_1$ and first right singular vector $v_1$. Recall that the *matrix norm* of $A$ is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.

**a)** Show that $\|Ax\|$ is maximized at $x = v_1$ (subject to $\|x\| = 1$), with maximum value $\sigma_1$.

**b)** Suppose now that $A$ is square and $\lambda$ is an eigenvalue of $A$. Show that $|\lambda| \leq \sigma_1$. (You may assume $\lambda$ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

**16.** **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

**b)** Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

**c)** Note that $A$ is very close to $A'$ numerically. Were the eigenvalues of $A$ close to the eigenvalues of $A'$? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are not*. This is another advantage of the SVD.

**17.** Decide if each statement is true or false, and explain why.

**a)** The left singular vectors of $A$ are eigenvectors of $A^TA$ and the right singular vectors are eigenvectors of $AA^T$.

**b)** For any matrix $A$, the matrices $AA^T$ and $A^TA$ have the same nonzero eigenvalues.

**c)** If $S$ is symmetric, then the nonzero eigenvalues of $S$ are its singular values.

**d)** If $A$ does not have full column rank, then 0 is a singular value of $A$.

**e)** Suppose that $A$ is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

**f)** The right singular vectors of $A$ are orthogonal to $\mathrm{Nul}(A)$.