Homework #2

due Monday, January 17, at 11:59pm

1. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries.

System of Equations	**Matrix Equation**	**Augmented Matrix**
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

2. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

3. Which of the following matrices are not in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \quad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

4. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

5. Which of the following matrices are not in reduced row echelon form? Why not?

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{pmatrix}$$

6. Describe all possible nonzero 2×2 matrices in RREF.

7. Use Gaussian elimination to reduce the following matrices into REF, and then Jordan substitution to reduce to RREF. Circle the first REF matrix that you produce, and circle the pivots in your REF and RREF matrices. You're welcome to use Rabinoff's Reliable Row Reducer.

$$\textbf{a)} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 \end{array}\right) \qquad \textbf{c)} \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \qquad \textbf{e)} \left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array}\right) \qquad \textbf{f)} \left(\begin{array}{ccccc|c} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array}\right)$$

8. Determine *how many* solutions each system of equations has. (Do not find the solutions.) [**Hint:** use Problem 7.]

$$\textbf{a)} \begin{cases} x_1 + x_2 \phantom{{}+x_3} = 1 \\ x_1 + x_2 + x_3 = 1 \\ x_2 + 2x_3 = 2 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 9x_3 = 1 \end{cases}$$

$$\textbf{c)} \begin{cases} 3x_2 - 6x_3 + 6x_4 + 4x_5 = -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 = 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 = 15 \end{cases}$$

9. Use Gaussian elimination and back-substitution to solve

$$\textbf{a)} \begin{cases} x_1 + x_2 \phantom{{}+x_3} = 1 \\ x_1 + 2x_2 + x_3 = 2 \\ x_2 + 2x_3 = 3 \end{cases} \qquad \textbf{b)} \begin{cases} x_1 + 3x_2 + 5x_3 = 7 \\ 3x_1 + 5x_2 + 7x_3 = 9 \\ 5x_1 + 7x_2 + 8x_3 = 1. \end{cases}$$

What happens if you replace 8 by 9 in (b)?

10. Three planes can fail to have an intersection point, even if no planes are parallel. Consider the two planes $A : x + y + z = 0$ and $B : x - 2y - z = 1$. Use the tool here:

https://technology.cpm.org/general/3dgraph/

to visualize these two planes, then answer the following questions:

a) What is the shape of the intersection $A \cap B$ of the two?

b) Use the equations of A and B to construct a third plane C whose intersection with the two is exactly the same as $A \cap B$. That is, $A \cap B \cap C = A \cap B$. [**Hint:** what happens if you add two equations together?]

c) Find a fourth plane D such that $A \cap D$, and $B \cap D$ are both non-empty, but $A \cap B \cap D$ is empty. That is, D should intersect both A and B, but the three should never meet. [**Hint:** make the system inconsistent!]

For both the last two parts, I strongly suggest you use the tool linked above to draw the planes and see your answers!

11. The parabola $y = ax^2 + bx + c$ passes through the points $(1,4)$, $(2,9)$, $(-1,6)$. Find the coefficients a, b, c.

12. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

13. Give examples of matrices A in *reduced row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?

14. Let A be a matrix in REF. Suppose that A has a pivot position in every *row*. Explain why the linear system $Ax = b$ is consistent. [**Hint:** What happens when you do back-substitution?]