Homework #3

due Monday, January 24, at 11:59pm

1. Use the formula for the 2×2 inverse to compute the inverses of the following matrices. If the matrix is not invertible, explain why.

$$\textbf{a)}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$$

2. Compute the inverses of the following matrices by Gauss–Jordan elimination. If the matrix is not invertible, explain why.

$$\textbf{a)}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \textbf{b)}\begin{pmatrix} 1 & 0 & -2 \\ 2 & -3 & 4 \\ -3 & 1 & 4 \end{pmatrix} \qquad \textbf{c)}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\textbf{d)}\begin{pmatrix} 6 & -4 & -7 & -1 \\ 7 & 0 & 1 & 3 \\ -1 & 2 & 3 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$$

3. Consider the linear system

$$\begin{aligned} x_1 + x_2 \phantom{{}+ x_3} &= b_1 \\ x_1 + 2x_2 + x_3 &= b_2 \\ x_2 + 2x_3 &= b_3. \end{aligned}$$

Use the Problem 2 to solve for x_1, x_2, x_3 in terms of b_1, b_2, b_3.

4. Decide if each statement is true or false, and explain why.

a) If A and B are invertible $n \times n$ matrices, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

b) If A is invertible then so is A^{10}.

c) An $n \times n$ matrix with n pivots is invertible.

d) An invertible $n \times n$ matrix has n pivots.

5. Suppose that

$$A\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad A\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

What is A^{-1}?

6. Consider a system of 3 equations in 4 variables. Write the elementary matrices that accomplish the following row operations:

a) $R_2 \mathrel{+}= 2R_1$

b) $R_1 \mathrel{-}= \frac{1}{2}R_3$

c) $R_3 \times= 2$

d) $R_3 \div= 2$

e) $R_1 \longleftrightarrow R_3$

f) $R_1 \longleftrightarrow R_2$

7. Consider a system of 3 equations in 4 variables. Write the row operations that the following elementary matrices perform on that system:

$$\textbf{a)} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

8. For each elementary matrix in Problem 7, write the row operation that un-does that row operation, and write its elementary matrix. Verify that this elementary matrix is the inverse of the matrix you started with. For instance:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{row op}} R_2 \mathrel{+}= R_1 \xrightarrow{\text{undo}} R_2 \mathrel{-}= R_1 \xrightarrow{\text{matrix}} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

9. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 1 & 3 & 1 \end{pmatrix}.$$

a) Explain how to reduce A to a matrix U in REF using three row replacements.

b) Let E_1, E_2, E_3 be the elementary matrices for these row operations, in order. Fill in the blank with a product involving the E_i:

$$U = \underline{\qquad\qquad} A.$$

c) Fill in the blank with a product involving the E_i^{-1}:

$$A = \underline{\qquad\qquad} U$$

d) Evaluate that product to produce a lower-triangular matrix L with ones on the diagonal such that $A = LU$.

e) Explain how to reduce U to the 3×3 identity matrix using three more row operations E_4, E_5, E_6.

f) Fill in the blank with a product involving the E_i:

$$A^{-1} = \underline{\hspace{3cm}}.$$

10. Solve the following matrix equations by forward- and back-substitution, using the provided LU decomposition. Check your answers by evaluating Ax.

a)
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} x = \begin{pmatrix} 14 \\ -26 \\ -16 \end{pmatrix}$$
$$\begin{pmatrix} 3 & 2 & 7 \\ -6 & -5 & -10 \\ -3 & 0 & -13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 7 \\ 0 & -1 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

b)
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} x = \begin{pmatrix} 3 \\ -4 \\ 10 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 4 & -3 & 2 \\ -2 & -7 & 7 & -7 \\ 4 & 17 & -17 & 19 \\ 2 & 4 & -5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -3 & 1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & -3 & 2 \\ 0 & -3 & 4 & -5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

c)
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} x = \begin{pmatrix} 2 \\ -3 \\ -21 \end{pmatrix}$$
$$\begin{pmatrix} 2 & 3 & -1 \\ 4 & 4 & 3 \\ 6 & 1 & 16 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & -1 \\ 0 & -2 & 5 \\ 0 & 0 & -1 \end{pmatrix}$$

11. Compute the $A = LU$ decomposition of the following matrices using the 2-column method. Check your answers by multiplying LU.

a) $\begin{pmatrix} 2 & 3 & 4 \\ -2 & 0 & -2 \\ -6 & -15 & -17 \end{pmatrix}$ **b)** $\begin{pmatrix} 3 & 0 & 2 & -1 \\ -6 & -1 & 1 & 3 \\ 6 & -4 & 26 & 5 \end{pmatrix}$ **c)** $\begin{pmatrix} 2 & 3 & 1 & 4 \\ -6 & -11 & -4 & -7 \\ -4 & -4 & -4 & -4 \\ 4 & 12 & -1 & 13 \end{pmatrix}$

12. Solve the following matrix equations by forward- and back-substitution, using the provided $PA = LU$ decomposition. Check your answers by evaluating Ax.

a)

$$\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} x = \begin{pmatrix} 54 \\ -59 \\ -14 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 20 & -19 & -5 \\ -20 & 19 & 0 \\ -5 & 4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -4 & -1 & 1 \end{pmatrix}\begin{pmatrix} -5 & 4 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} x = \begin{pmatrix} 12 \\ 4 \\ 0 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 3 & -3 & 1 & 0 \\ 0 & -4 & -5 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -2 & -1 \\ 0 & -2 & 3 & 0 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

13. Compute a $PA = LU$ decomposition for each of the following matrices, using the 3-column method and performing *maximal partial pivoting*. Check your answers by multiplying PA and LU.

a) $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ -1 & 1 & 1 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 2 & 5 & 0 \\ 1 & 2 & 4 & 2 \\ 0 & -1 & 0 & 8 \\ -1 & -3 & -1 & -1 \end{pmatrix}$

14. Recall that a *permutation matrix* is a product of elementary matrices for row swaps.

a) If P is the $n \times n$ elementary matrix for a row swap, explain why $P^{-1} = P = P^T$.

b) If P is any permutation matrix, show that $P^{-1} = P^T$. [**Hint:** write $P = P_1 P_2 \cdots P_r$ for elementary row swaps P_i.] Is $P = P^T$ for a general permutation matrix?

c) Explain why a permutation matrix has exactly one 1 in each row and in each column, with all other entries equal to zero. Is any matrix of this form a permutation matrix? Why or why not?

d) Let A be an $m \times n$ matrix and let P be the $n \times n$ elementary matrix for swapping row i and row j. Show that AP is the matrix obtained from A by swapping *columns* i and j. [**Hint:** $(AP)^T = PA^T$.]

15. Consider the matrix

$$A = \begin{pmatrix} 0 & 8 & -17 & 28 \\ 1 & -2 & -2 & -1 \\ -1 & 0 & 5 & 1 \\ 3 & 0 & -14 & -8 \end{pmatrix}.$$

In this problem, we will see how to produce a $PA = LU$ decomposition of A using elementary matrices.

a) Perform Gaussian elimination on A using maximal partial pivoting. Write down the elementary matrices you use. You should end up with

$$U = \begin{pmatrix} 3 & 0 & -14 & -8 \\ 0 & 8 & -17 & 28 \\ 0 & 0 & -\frac{19}{12} & \frac{26}{3} \\ 0 & 0 & 0 & \frac{3}{19} \end{pmatrix} = E_4P_3E_3P_2E_2E_1P_1A$$

where P_1, P_2, P_3 are elementary matrices for row swaps and E_1, E_2, E_3, E_4 correspond to row replacements.

b) Explain why $P_2E_2E_1 = (P_2E_2E_1P_2)P_2$. Compute $P_2E_2E_1P_2$, and verify that it is lower-unitriangular. (Do not multiply matrices: perform row operations!)

c) Compute $P_3E_3(P_2E_2E_1P_2)P_3$ using (b) and using Problem 14(d) to multiply on the left and right by permutation matrices. Verify that it is lower-unitriangular.

d) Rewrite $U = E_4P_3E_3P_2E_2E_1P_1A$ as

$$U = E_4\big(P_3E_3(P_2E_2E_1P_2)P_3\big)P_3P_2P_1A$$

Which products of elementary matrices are L^{-1} and P?