Homework #4

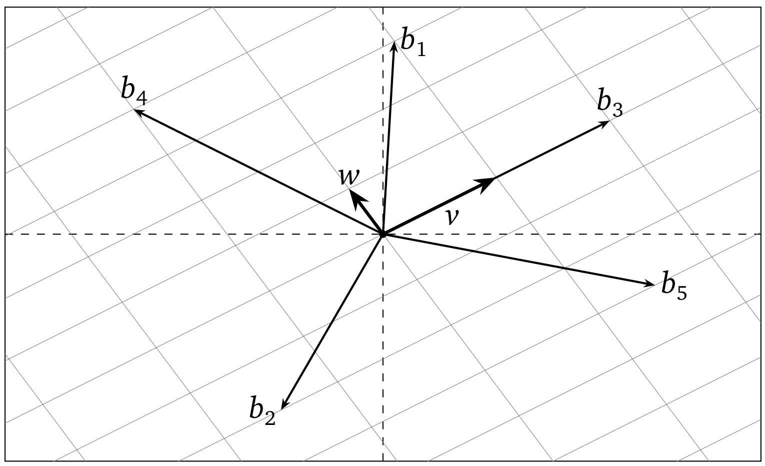
due Monday, January 31, at 11:59pm

1. Consider the vectors

$$v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad w = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Draw the 16 linear combinations $cv + dw$ $(c, d = -1, 0, 1, 2)$ as *points* in the xy-plane.

2. Certain vectors v, w in $\mathbf{R}^2$ are drawn below. Express each of b_1, b_2, b_3, b_4, b_5 as a linear combination of v, w. *Do not try to guess the coordinates of v and w!*



3. If

$$v + w = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \quad \text{and} \quad v - w = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

compute and draw the vectors v and w.

4. Consider the vectors

$$u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Draw a picture of all of the linear combinations $au + bv$ for real numbers a, b satisfying $0 \le a \le 1$ and $0 \le b \le 1$. (This will be a shaded region in the xy-plane.)

5. For each matrix A and vector b, decide if the system $Ax = b$ is consistent. If so, find the parametric form of the general solution of $Ax = b$. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad x_1 = x_2 + 1.$$

Also answer the following questions: Which variables are free? How many solutions does the system have?

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 48 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

e) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

6. For each matrix A and vector b in Problem 5, find the parametric *vector* form of the general solution of $Ax = b$ (if the system is consistent). For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

What is the dimension of the solution set?

7. The equation $x + 2y = z$ determines a plane in $\mathbf{R}^3$. (This is an *implicit equation* for the plane).

a) What is the coefficient matrix A for this system?

b) Which are the free variables?

c) Write the parametric form of the solutions of $x + 2y = z$. This expresses the points on the plane in terms of two *parameters*.

d) Do the same for the plane defined by $2y = z$. What is different?

8. The equations

$$\begin{aligned} x + y + z &= 0 \\ x - 2y - z &= 1 \end{aligned}$$

determine a line $\mathbf{R}^3$. (These are *implicit equations* for the line). Write the line in parameterized form: that is, find three linear functions $f_1(t), f_2(t), f_3(t)$ in one variable such that all points on the line have the form $(x, y, z) = (f_1(t), f_2(t), f_3(t))$ for a unique value of t. (Use the free variable as the parameter t.)

9. Find a 2×3 matrix A in RREF and a vector b such that the solution set of $Ax = b$ consists of all vectors of the form

$$\begin{pmatrix} 1+t \\ 2-t \\ t \end{pmatrix} \qquad t \in \mathbf{R}.$$

10. Decide if each statement is true or false, and explain why.

a) A square matrix has no free variables.

b) An invertible matrix has no free variables.

c) An $m \times n$ matrix has at most m pivots.

d) A wide matrix (more columns than rows) must have a free variable.

e) If A is a tall matrix (more rows than columns), then $Ax = b$ has at most one solution.

11. Express each system of linear equations as a vector equation. For example,

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ -x_1 - x_2 &= 4 \end{aligned} \quad \rightsquigarrow \quad x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

a) $\begin{cases} 3x_1 + 2x_2 + 4x_3 = 9 \\ -x_1 + 4x_3 = 2 \end{cases}$ **b)** $\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

c) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

12. **a)** Is $\begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}$?

If so, what are the weights?

b) Find a vector that is *not* a linear combination of the columns of the matrix

$$\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}.$$

[**Hint:** for both parts, compare Problem 5.]

13. For each matrix A and vector b, and express the solution set in the form

$$p + \mathrm{Span}\{???\}$$

for some vector p. For instance,

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

[**Hint:** You found the parametric vector form in Problem 6.]

a) $$A = \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

b) $$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 2 \\ 49 \end{pmatrix}$$

c) $$A = \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 4 \\ -6 \\ 10 \end{pmatrix}$$

d) $$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$$

14. For each matrix A in Problem 13, write the solution set of $Ax = 0$ as a span. Does there exist a nontrivial solution? Do not do Gauss–Jordan elimination again!

15. Suppose that A is a 3×3 matrix and b is a vector such that $Ax = b$ is a line in $\mathbf{R}^3$. How many pivots does A have?

16. When is the following system consistent?

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &= b_1 \\ -4x_1 - 5x_2 + 5x_3 &= b_2 \\ 6x_1 + x_2 + 12x_3 &= b_3 \end{aligned}$$

Your answer should be a single linear equation in b_1, b_2, b_3. [**Hint:** perform Gaussian elimination.] Explain the relationship between your answer and

$$\mathrm{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}.$$

17. Let A be a 3×4 matrix whose columns span the plane $x + y + z = 0$.

a) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *consistent*.

b) Find a vector $b \in \mathbf{R}^3$ making the system $Ax = b$ *inconsistent*.

18. Draw a picture of all vectors $b \in \mathbf{R}^2$ for which the equation

$$\begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix} x = b$$

is consistent. [**Hint:** the answer is a span!]

19. Suppose that A is a 2×3 matrix such that

$$A\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

a) Find two different solutions of $Ax = 0$.

b) Find two more solutions of $Ax = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

20. Suppose that $Ax = b$ is consistent. Explain why $Ax = b$ has a unique solution precisely when $Ax = 0$ has only the trivial solution.

21. Give geometric descriptions of the following spans (line, plane, ...).

a) $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}\right\}$ **b)** $\operatorname{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right\}$ **c)** $\operatorname{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ -6 \end{pmatrix}\right\}$

d) $\operatorname{Span}\left\{\begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 5 \\ 12 \end{pmatrix}\right\}$ **e)** $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}\right\}$

[**Hint:** for **d)**, compare Problem 16.]

22. **a)** List five nonzero vectors contained in $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}$.

b) Is $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ contained in $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}\right\}$?

If so, express $\begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

c) Show that $\begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$ is contained in $\operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}\right\}$.

d) Describe $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$ geometrically.

e) Find a vector not contained in $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$.

23. Decide if each statement is true or false, and explain why.

a) A vector b is a linear combination of the columns of A if and only if $Ax = b$ has a solution.

b) There is a matrix A such that $Ax = \binom{2}{2}$ has infinitely many solutions and $Ax = \binom{2}{-2}$ has exactly one solution.

c) The zero vector is contained in every span.

d) The matrix equation $Ax = 0$ can be consistent or inconsistent, depending on what A is.

e) If the zero vector is a solution of a system of equations, then the system is homogeneous.

f) If $Ax = b$ has a unique solution, then A has a pivot in every column.

g) If $Ax = b$ is consistent, then the solution set of $Ax = b$ is obtained by translating the solution set of $Ax = 0$.

h) It is possible for $Ax = b$ to have exactly 13 solutions.