

Homework #5

due **Wednesday**, February 9, at 11:59pm

1. Find a basis for the null space of each matrix, and express the null space as the column space of some other matrix.

$$\text{a) } \begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

$$\text{c) } \begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix} \quad \text{d) } \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

[Hint: Compare HW4#14.]

2. Draw pictures of the null space and the column space of the following matrices. Be precise!

a) $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$:

	Nul(A)	Col(A)

b) $A = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$:

	Nul(A)	Col(A)

c) $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$:

	Nul(A)	Col(A)

3. Give examples of subsets V of \mathbf{R}^2 such that:
- a) V is closed under addition and contains 0, but is not closed under scalar multiplication.

b) V is closed under scalar multiplication and contains 0 , but is not closed under addition.

c) V is closed under addition and scalar multiplication, but does not contain 0 .

Therefore, none of these conditions is redundant.

4. Which of the following subsets of \mathbf{R}^3 are subspaces? If it is not a subspace, why not? If it is, write it as the column space or null space of some matrix.

a) The plane $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) The plane $\{(x, y, 1) : x, y \in \mathbf{R}\}$.

c) The set consisting of all vectors (x, y, z) such that $xy = 0$.

d) The set consisting of all vectors (x, y, z) such that $x \leq y$.

e) The span of $(1, 2, 3)$ and $(2, 1, -3)$.

f) The solution set of the system of equations
$$\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$$

g) The solution set of the system of equations
$$\begin{cases} x + y + z = 0 \\ x - 2y - z = 1. \end{cases}$$

5. Give a geometric description of the following column spaces (line, plane, ...).

a) $\text{Col} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ b) $\text{Col} \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix}$ c) $\text{Col} \begin{pmatrix} 0 & 0 \\ 1 & -2 \\ 3 & -6 \end{pmatrix}$

d) $\text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$ e) $\text{Col} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

[Hint: Compare HW4#21.]

6. Find a nonzero 2×2 matrix such that $A^2 = 0$.

7. a) Explain why $\text{Col}(AB)$ is contained in $\text{Col}(A)$.

b) Give an example where $\text{Col}(AB) \neq \text{Col}(A)$.

[Hint: use Problem 6.]

8. a) Explain why $\text{Nul}(AB)$ contains $\text{Nul}(B)$.

b) Give an example where $\text{Nul}(AB) \neq \text{Nul}(B)$.

[Hint: use Problem 6.]

9. Suppose that A is in reduced row echelon form. Describe $\text{Col}(A)$.

10. Decide if each statement is true or false, and explain why.

- a) The null space of an $m \times n$ matrix with n pivots is \mathbf{R}^n .
- b) If $\text{Col}(A) = \{0\}$, then A is the zero matrix.
- c) The column space of $2A$ equals the column space of A .
- d) The null space of $A + B$ contains the null space of A .
- e) If U is an echelon form of A , then $\text{Nul}(U) = \text{Nul}(A)$.
- f) If U is an echelon form of A , then $\text{Col}(U) = \text{Col}(A)$.

11. Which sets of vectors are linearly independent? If the vectors are linearly dependent, find a linear relation among them.

a) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$ b) $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$ c) $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$

d) $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ -5 \\ 2 \\ -7 \end{pmatrix}, \begin{pmatrix} -1 \\ 4 \\ -3 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 6 \end{pmatrix} \right\}$ e) $\left\{ \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\}$

Which sets do you know are linearly dependent without doing any work?

- 12.**
- a) For each set in Problem 11, find a basis for the span of the vectors.
 - b) For each set in Problem 11, find a *different* basis for the span of the vectors. Your new basis cannot contain a scalar multiple of any vector in your answer for a).
 - c) What is the dimension of each of these spans?

13. Consider the vectors

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}$$

of Problem 11(a).

- a) Find two different ways to express $(5, 7, 9)$ as a linear combination of these vectors.
 - b) How many ways can you express $(5, 7, 9)$ as a linear combination of the first two vectors?
- 14.** Let $\{w_1, w_2, w_3\}$ be a basis for a subspace V , and set

$$v_1 = w_2 + w_3 \quad v_2 = w_1 + w_3 \quad v_3 = w_1 + w_2.$$

Show that $\{v_1, v_2, v_3\}$ is also a basis for V .

15. Certain vectors v_1, v_2, v_3, v_4 span a 3-dimensional subspace of \mathbf{R}^5 . They satisfy the linear relation

$$2v_1 + 0v_2 - v_3 + v_4 = 0.$$

- a) Describe *all* linear relations among v_1, v_2, v_3, v_4 .
[Hint: what is the rank of the matrix with columns v_1, v_2, v_3, v_4 ?]
b) Which vector is *not* in the span of the others, and why?

16. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

Which sets of columns form a basis for the column space? (I.e., do the first and third columns form a basis? what about the second and third? etc.)

17. Find bases for the following subspaces.

- a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.
b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.
c) The solution set of the system of equations $\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$
d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.
e) The subspace of all vectors in \mathbf{R}^3 whose coordinates sum to zero.
f) The intersection of the plane $x - 2y - z = 0$ with the xy -plane.

18. Let V be a 4-dimensional subspace of \mathbf{R}^5 .

- a) Explain why every basis for V can be extended to a basis for \mathbf{R}^5 by adding one more vector.
b) Find an example of a 4-dimensional subspace V of \mathbf{R}^5 and a basis for \mathbf{R}^5 that cannot be reduced to a basis for V by removing one vector.

19. Decide if each statement is true or false, and explain why.

- a) If v_1, v_2, \dots, v_n are linearly independent vectors, then $\text{Span}\{v_1, v_2, \dots, v_n\}$ has dimension n .
b) If the matrix equation $Ax = 0$ has the trivial solution, then the columns of A are linearly independent.
c) If $\text{Span}\{v_1, v_2\}$ is a plane and the set $\{v_1, v_2, v_3\}$ is linearly dependent, then $v_3 \in \text{Span}\{v_1, v_2\}$.

- d)** If v_3 is not a linear combination of v_1 and v_2 , then $\{v_1, v_2, v_3\}$ is linearly independent.
- e)** If $\{v_1, v_2, v_3\}$ is linearly dependent, then so is $\{v_1, v_2, v_3, x\}$ for any vector x .
- f)** The set $\{0\}$ is linearly independent.
- g)** If $\{v_1, v_2, v_3, v_4\}$ is linearly independent, then so is $\{v_1, v_2, v_3\}$.
- h)** The columns of any 4×5 matrix are linearly dependent.
- i)** If $Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ has only one solution, then the columns of A are linearly independent.
- j)** If $\text{Span}\{v_1, v_2, v_3\}$ has dimension 3, then $\{v_1, v_2, v_3\}$ is linearly independent.