Homework #6

due Monday, February 14, at 11:59pm

1. Find bases for the four fundamental subspaces of each matrix, and compute their dimensions. Verify that $\dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$ is the number of columns of A, that $\dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T)$ is the number of rows, and that $\dim \mathrm{Row}(A) = \dim \mathrm{Col}(A)$.

[**Hint:** Augment with the $m \times m$ identity matrix so you only have to do Gauss–Jordan elimination once.]

a) $\begin{pmatrix} 2 & 1 & 1 & 4 \\ 4 & 2 & 1 & 7 \end{pmatrix}$ **b)** $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$ **c)** $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

d) $\begin{pmatrix} 1 & 2 & 3 & -1 & 1 \\ -2 & -4 & -5 & 4 & 1 \\ 1 & 2 & 2 & -3 & -1 \\ -3 & -6 & -7 & 7 & 6 \end{pmatrix}$ **e)** $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

2. Suppose that A is an invertible 4×4 matrix. Find bases for its four fundamental subspaces.

3.
a) Let A be a 9×4 matrix of rank 3. What are the dimensions of its four fundamental subspaces?

b) If the left null space of a 5×4 matrix A has dimension 3, what is the rank of A?

4. Find an example of a matrix with the required properties, or explain why no such matrix exists.

a) The column space contains $(1, 2, 3)$ and $(4, 5, 6)$, and the row space contains $(1, 2)$ and $(2, 3)$.

b) The column space has basis $\{(1, 2, 3)\}$, and the null space has basis $\{(3, 2, 1)\}$.

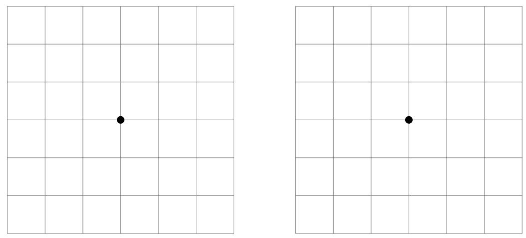
c) The dimension of the null space is one greater than the dimension of the left null space.

d) A 3×5 matrix whose row space equals its null space.

5. Draw the four fundamental subspaces of the following matrices, in grids like below.

a) $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$

Row Picture *Column Picture*



6. For the following matrix A, compute the reduced row echelon form of A and of A^T. Do they have the same free variables? Do they have the same rank?

$$A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 4 & 5 & 6 \end{pmatrix}$$

7. Find a matrix A such that

$$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

What is the rank of A?

8. **a)** If $\mathrm{Col}(B)$ is contained in $\mathrm{Nul}(A)$, then $AB =$ __________.

b) Find a 2×2 matrix A such that $\mathrm{Col}(A) = \mathrm{Nul}(A)$. What is the rank of such a matrix? [**Hint:** use HW5#6.]

9. Let A be a 3×3 matrix of rank 2. Explain why A^2 is not the zero matrix.
[**Hint:** Compare Problem 8.]

10. **a)** Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$. [**Hint:** Compare HW5#7.]

b) Show that $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$. [**Hint:** Take transposes.]

11. This problem explains why we only consider *square* matrices when we discuss invertibility.

a) Show that a tall matrix A (more rows than columns) does not have a right inverse, i.e., there is no matrix B such that $AB = I_m$.

b) Show that a wide matrix A (more columns than rows) does not have a left inverse, i.e., there is no matrix B such that $BA = I_n$.

[**Hint:** compare Problem 10.]

12. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full column rank"?

a) $\mathrm{Nul}(A) = \{0\}$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Row}(A) = n$

e) The columns of A span $\mathbf{R}^m$

f) A^T has full column rank

13. Let A be an $m \times n$ matrix. Which of the following are *equivalent* to the statement "A has full row rank"?

a) $\mathrm{Col}(A) = \mathbf{R}^m$

b) A has rank m

c) The columns of A are linearly independent

d) $\dim \mathrm{Nul}(A) = n - m$

e) The rows of A span $\mathbf{R}^n$

f) A^T has full column rank

14. Consider the following vectors:

$$u = \begin{pmatrix} -.6 \\ .8 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ 3 \end{pmatrix} \quad w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

a) Compute the lengths $\|u\|, \|v\|$, and $\|w\|$.

b) Compute the lengths $\|2u\|, \|-v\|$, and $\|3w\|$.

c) Find the unit vectors in the directions of u, v, and w.

d) Compute the dot products $u \cdot v, u \cdot w$, and $v \cdot w$. Verify that they are the same as $v \cdot u, w \cdot u$, and $w \cdot v$, respectively.

e) Check the Schwartz inequalities $|u \cdot v| \leq \|u\| \, \|v\|$ and $|v \cdot w| \leq \|v\| \, \|w\|$.

f) Find the angles between u and v and between v and w.

g) Find the distance from v to w.

h) Find unit vectors u', v', w' orthogonal to u, v, w, respectively.

15. What is the length of the vector $v = (1, 1, \ldots, 1)$ in n dimensions?

16. If $\|v\| = 5$ and $\|w\| = 3$, what are the smallest and largest possible values of $\|v - w\|$? What are the smallest and largest possible values of $v \cdot w$? Justify your answer using the algebra of dot products.

17. **a)** If $v \cdot w < 0$, what does that say about the angle between v and w?

b) Find three vectors u, v, w in the xy-plane such that $u \cdot v < 0$, $u \cdot w < 0$, and $v \cdot w < 0$.

18. Compute a basis for the orthogonal complement of each of the following spans.

a) $\text{Span}\left\{\begin{pmatrix}1\\2\\-1\end{pmatrix}\right\}$ **b)** $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\right\}$ **c)** $\text{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$

d) $\text{Span}\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ **e)** $\text{Span}\{\} = \{0\}$

f) $\text{Span}\left\{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}\right\}$