

Homework #7

due Monday, February 21, at 11:59pm

1. Compute a basis for the orthogonal complement of each the following subspaces.

$$\text{a) Col} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{b) Nul} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \quad \text{c) Row} \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

$$\text{d) Nul} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \quad \text{e) Span} \left\{ \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \right\} \quad \text{f) Col} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \end{pmatrix}$$

[Hint: solving a)–d) requires only one Gauss-Jordan elimination, and f) doesn't require any work.]

2. Compute a basis for the orthogonal complement of each the following subspaces.

a) $\{(x, y, x) : x, y \in \mathbf{R}\}$.

b) $\{(x, y, z) \in \mathbf{R}^3 : x = 2y + z\}$.

c) The solution set of the system of equations
$$\begin{cases} x + y + z = 0 \\ x - 2y - z = 0. \end{cases}$$

d) $\{x \in \mathbf{R}^3 : Ax = 2x\}$, where $A = \begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$.

e) The subspace of all vectors in \mathbf{R}^3 whose coordinates sum to zero.

f) The intersection of the plane $x - 2y - z = 0$ with the xy -plane.

g) The line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[Hint: Compare HW5#17.]

3. For each pair of vectors v and w , draw $\text{Span}\{v\}$, and compute and draw the projection p of w onto $\text{Span}\{v\}$.

$$\text{a) } v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \quad \text{b) } v = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(Here θ is any number, so w is just a vector on the unit circle.)

4. For each subspace V and vector b , compute the orthogonal projection b_V of b onto V by solving a normal equation $A^T A x = A^T b$, and find the distance from b to V .

a)
$$V = \text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$$

$$\text{b) } V = \text{Col} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \quad b = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$$

$$\text{c) } V = \text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \quad b = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$$

5. For each subspace V , compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ of the vector $b = (1, 2, -1)$ with respect to V .

$$\text{a) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} \quad \text{b) } V = \text{Nul} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

$$\text{c) } V = \mathbf{R}^3 \quad \text{d) } V = \{0\}$$

[Hint: Only part a) requires any work.]

6. Compute the orthogonal decomposition $(3, 1, 3) = b_V + b_{V^\perp}$ with respect to each subspace of V of Problem 1(a)–(e).

[Hint: Only parts a) and c) require any work, and even c) doesn't require work if you're clever enough. In fact, you can solve all five parts by computing two dot products.]

7. Let A be an $m \times n$ matrix, and let $b \in \mathbf{R}^m$ be a vector. Suppose that $A^T b = 0$. Compute the orthogonal decomposition $b = b_V + b_{V^\perp}$ with respect to $V = \text{Col}(A)$.

8. a) Find an implicit equation for the plane

$$\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} \right\}.$$

[Hint: use Problem 1(a).]

- b) Find implicit equations for the line $\{(t, -t, t) : t \in \mathbf{R}\}$.

[Hint: use Problem 2(g).]

9. Construct a matrix A with each of the following properties, or explain why no such matrix exists.

a) The column space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

b) The row space contains $\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$, and the null space contains $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$.

c) $Ax = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is consistent, and $A^T \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = 0$.

d) A 2×2 matrix A with no zero entries such that every row of A is orthogonal to every column.

e) The sum of the columns of A is $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, and the sum of the rows of A is $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

10. Suppose that S is a *symmetric matrix*: that is, a square matrix such that $S = S^T$. Explain why $\text{Col}(S)$ is orthogonal to $\text{Nul}(S)$.

11. Show that $A^T A = 0$ is only possible when $A = 0$.

12. Construct a 3×3 matrix A , with no zero entries, whose columns are orthogonal to each other. Compute $A^T A$, and explain why this matrix is diagonal.

13. Explain why A has full column rank if and only if $A^T A$ is invertible.

14. Decide if each statement is true or false, and explain why.

a) Two subspaces that meet only at the zero vector are orthogonal.

b) If A is a 3×4 matrix, then $\text{Col}(A)^\perp$ is a subspace of \mathbf{R}^4 .

c) If A is any matrix, then $\text{Nul}(A) = \text{Nul}(A^T A)$.

d) If A is any matrix, then $\text{Row}(A) = \text{Row}(A^T A)$.

e) If every vector in a subspace V is orthogonal to every vector in another subspace W , then $V = W^\perp$.

f) If x is in V and V^\perp , then $x = 0$.

g) If x is in a subspace V , then the orthogonal projection of x onto V is x .

h) If x is in the orthogonal complement of a subspace V , then the orthogonal projection of x onto V is x .

15. For each column space V , compute the projection matrix P_V . Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\begin{aligned} \text{a) } V &= \text{Col} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} & \text{b) } V &= \text{Col} \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} \\ \text{c) } V &= \text{Col} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} \end{aligned}$$

16. For each subspace V , compute the projection matrix P_V . Verify that $P_V^2 = P_V$ and that $P_V^T = P_V$.

$$\text{a) } V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\} \quad \text{b) } V = \text{Nul} \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{pmatrix}$$

17. For each vector v , compute the projection matrix onto $V = \text{Span}\{v\}$ using the formula $P_V = vv^t/v \cdot v$.

$$\text{a) } v = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{b) } v = \begin{pmatrix} 3 \\ 0 \\ 4 \\ -1 \end{pmatrix} \quad \text{c) } v = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \text{ (in } \mathbf{R}^n \text{)}$$

18. a) Compute P_V for $V = \mathbf{R}^n$.
b) Compute P_V for $V = \{0\}$.

19. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 3 \end{pmatrix},$$

and let $V = \text{Col}(A)$.

- a) Compute P_V using the formula $P_V = A(A^T A)^{-1} A^T$.
b) Compute a basis $\{v_1, v_2\}$ for $V^\perp = \text{Nul}(A^T)$.
c) Let B be the matrix with columns v_1, v_2 , and compute P_{V^\perp} using the formula $B(B^T B)^{-1} B^T$.
d) Verify that your answers to (a) and (c) sum to I_4 .

(Factor out $ad - bc$ and use a computer to do the matrix multiplication! Your answers should be in fractions, not decimals.)

This illustrates the fact that once you've computed P_V , there's no need to compute P_{V^\perp} separately. It's a lot of extra work!

20. Consider the plane V defined by the equation $x + 2y - z = 0$. Compute the matrix P_V for orthogonal projection onto V in two ways:

a) Find a basis for V , put your basis vectors into a matrix A , and use the formula $P_V = A(A^T A)^{-1} A^T$.

b) Compute the matrix for orthogonal projection P_{V^\perp} onto the line V^\perp using the formula $v v^T / v \cdot v$, and subtract: $P_V = I_3 - P_{V^\perp}$.

[**Hint:** It doesn't take any work to find a basis for V^\perp .]

If V is defined by a single equation in 1 000 000 variables, which method do you think a computer would be able to implement?

21. Compute the matrices P_1, P_2 for orthogonal projection onto the lines through $a_1 = (-1, 2, 2)$ and $a_2 = (2, 2, -1)$, respectively. Now compute $P_1 P_2$, and explain why it is what it is.

22. Decide if each statement is true or false, and explain why. In each statement, V is a subspace of \mathbf{R}^n .

a) The rank of P_V is equal to $\dim(V)$.

b) $P_V P_{V^\perp} = 0$.

c) $P_V + P_{V^\perp} = 0$.

d) $\text{Col}(P_V) = V$.

e) $\text{Nul}(P_V) = V$.

f) $\text{Row}(P_V) = \text{Col}(P_V)$.

g) $\text{Nul}(P_V)^\perp = \text{Col}(P_V)$.