Homework #9

due Wednesday, March 16, at 11:59pm

1. Compute the determinants of the following matrices *using Gaussian elimination*.

$$\textbf{a)} \begin{pmatrix} -2 & 1 \\ 1 & 3 \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix}$$

$$\textbf{c)} \begin{pmatrix} -4 & -3 & -3 & -2 \\ 4 & 1 & 2 & -2 \\ -12 & -3 & -9 & 3 \\ 0 & 8 & 19 & 33 \end{pmatrix} \qquad \textbf{d)} \begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix}$$

2. Suppose that

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 10 \quad \text{and} \quad \det\begin{pmatrix} a' & b' & c' \\ d & e & f \\ g & h & i \end{pmatrix} = 5.$$

Find the determinants of the following matrices.

$$\textbf{a)} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \qquad \textbf{b)} \begin{pmatrix} a & b & c \\ d & e & f \\ g+2d & h+2e & i+2f \end{pmatrix} \qquad \textbf{c)} \begin{pmatrix} a & b & c \\ \frac{1}{2}d & \frac{1}{2}e & \frac{1}{2}f \\ g & h & i \end{pmatrix}$$

$$\textbf{d)} \begin{pmatrix} g & h & i \\ a & b & c \\ d & e & f \end{pmatrix} \qquad \textbf{e)} \begin{pmatrix} a & b & c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix} \qquad \textbf{f)} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

$$\textbf{g)}\ 2\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \textbf{h)} \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \textbf{i)} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{-1}$$

$$\textbf{j)}\ -\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \qquad \textbf{k)} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^{3} \qquad \textbf{l)} \begin{pmatrix} a & b+2c & c \\ d & e+2f & f \\ g & h+2i & i \end{pmatrix}$$

$$\textbf{m)} \begin{pmatrix} a+2a' & b+2b' & c+2c' \\ d & e & f \\ g & h & i \end{pmatrix}$$

3. Compute

$$\det\left[\begin{pmatrix} -3 & 3 & 2 \\ 3 & 0 & 0 \\ -9 & 18 & 7 \end{pmatrix} - \lambda I_3\right]$$

where λ is an unknown real number. Your answer will be a function of λ.

4. Find $\det(E)$ when:

a) E is the elementary matrix for a row replacement.

b) E is the elementary matrix for $R_i \times = c$.

c) E is the elementary matrix for a row swap.

5. A matrix A has the $PA = LU$ factorization

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} A = L \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & -3 \end{pmatrix}.$$

What is $\det(A)$?.

6. **a)** Compute the determinants of the matrices in Problem 1 in two more ways: by expanding cofactors along a row, and by expanding cofactors along a column. You should get the same answer using all three methods!

b) Compute the determinants of the matrices in Problem 1(b) and (d) *again* using Sarrus' scheme.

c) For the matrix of Problem 1(c), sum the products of the forward diagonals and subtract the products of the backward diagonals, as in Sarrus' scheme. Did you get the determinant?

7. Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

a) Compute the cofactor matrix C of A.

b) Compute AC^T. What is the relationship between C^T and A^{-1}?

8. Consider the $n \times n$ matrix F_n with 1's on the diagonal, 1's in the entries immediately below the diagonal, and −1's in the entries immediately above the diagonal:

$$F_2 = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad F_3 = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \qquad F_4 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \cdots.$$

a) Show that $\det(F_2) = 2$ and $\det(F_3) = 3$.

b) Expand in cofactors to show that $\det(F_n) = \det(F_{n-1}) + \det(F_{n-2})$.

c) Compute $\det(F_4), \det(F_5), \det(F_6), \det(F_7)$ using **b)**.

This shows that $\det(F_n)$ is the nth *Fibonacci number*. (The sequence usually starts with $1, 1, 2, 3, \ldots$, so our $\det(F_n)$ is the usual $n+1$st Fibonacci number.)

9. Let A be an $n \times n$ invertible matrix with integer (whole number) entries.

a) Explain why $\det(A)$ is an integer.

b) If $\det(A) = \pm 1$, show that A^{-1} has integer entries.

c) If A^{-1} has integer entries, show that $\det(A) = \pm 1$.

10. Recall that an *orthogonal matrix* is a square matrix with orthonormal columns, or equivalently, a square matrix Q such that $Q^T Q = I_n$.

a) Prove that every orthogonal matrix has determinant ± 1.

b) Prove that the cofactor matrix of an orthogonal matrix Q is $\pm Q$.

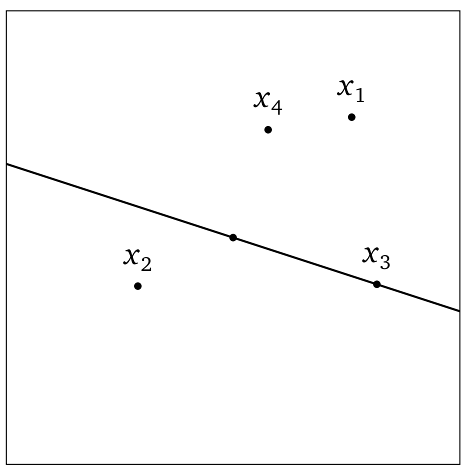
c) Show that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ is orthogonal, and compute its determinant.

11. Let V be a subspace of $\mathbf{R}^n$. The matrix for *reflection over* V is

$$R_V = I_n - 2P_{V^\perp},$$

where $P_{V^\perp} = I_n - P_V$ is the projection matrix onto $V^\perp$.

a) Suppose that V is the line in the picture. Draw the vectors $R_V x_1, R_V x_2, R_V x_3$, and $R_V x_4$ as points in the plane.



b) Show that any reflection matrix R_V is orthogonal.
[**Hint:** Recall that $P_{V^\perp}^2 = P_{V^\perp} = P_{V^\perp}^T$.]

c) Let V be the plane $x + y + z = 0$. Compute R_V and $\det(R_V)$.

d) Let V be any plane in $\mathbf{R}^3$. Prove that $\det(R_V) = -1$, as follows: choose an orthonormal basis $\{u_1, u_2\}$ for V, and let $u_3 = u_1 \times u_2$. Show that the matrix A with columns u_1, u_2, u_3 has determinant 1, and that $R_V A$ has determinant -1.

Summary: a reflection over a plane in $\mathbf{R}^3$ has determinant -1.

e) Now compute $\det(R_L)$, where L is the x-axis in $\mathbf{R}^3$.

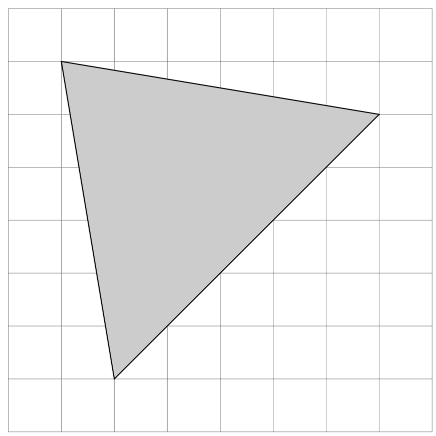
12. Let V be a subspace of $\mathbf{R}^n$ and let P_V be the projection matrix onto V.

a) Find $\det(P_V)$ when $V \neq \mathbf{R}^n$.

b) Find $\det(P_V)$ when $V = \mathbf{R}^n$.

13. Let C be the *hypercube* in $\mathbf{R}^4$ with corners $(\pm1, \pm1, \pm1, \pm1)$. Compute the volume of C.

14. Compute the area of the triangle pictured below using a 2×2 determinant. (The grid marks are one unit apart.)



15. Use a cross product to find an implicit equation for the plane

$$V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} \right\}.$$

Compare HW7#8(a).

16. **a)** Let $v = (a, b)$ and $w = (c, d)$ be vectors in the plane, and let $A = \left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$. By taking the cross product of $(a, b, 0)$ and $(c, d, 0)$, explain how the right-hand rule determines the sign of $\det(A)$.

b) Using the identity

$$\left[\begin{pmatrix} a \\ b \\ c \end{pmatrix} \times \begin{pmatrix} d \\ e \\ f \end{pmatrix} \right] \cdot \begin{pmatrix} g \\ h \\ i \end{pmatrix} = \det \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix},$$

explain how the right-hand rule determines the sign of a 3×3 determinant.

17. Decide if each statement is true or false, and explain why.

a) $\det(A+B) = \det(A) + \det(B)$.

b) $\det(ABC^{-1}) = \dfrac{\det(A)\det(B)}{\det(C)}$.

c) $\det(AB) = \det(BA)$.

d) $\det(3A) = 3\det(A)$.

e) If A^5 is invertible then A is invertible.

f) The determinant of A is the product of its diagonal entries.

g) If the columns of A are linearly dependent, then $\det(A) = 0$.

h) The determinant of the cofactor matrix of A equals the determinant of A.

i) If A is a 3×3 matrix with determinant zero, then two of the columns of A are scalar multiples of each other.

j) $u \times v = v \times u$.

k) If $u \times v = 0$ then $u \perp v$.