1.11.22

Solving Systems of Linear Equations with Elimination

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$
$$3x_1 + x_2 - x_3 = -2$$

How to solve it?

Substitution: solve 1 eq. for x_1, plug into 2^{nd} + 3^{rd} eq., simplify + continue

Elimination: combine equations to eliminate variables.

↑ We focus on this b/c it scales better to lots of eqs. + variables.

Example:

$$\boxed{\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}}$$

$\leftarrow$ replace R_2 by $R_2 - 2R_1$

$R_2 -= 2R_1$ $\rightsquigarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 0 - 7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$\uparrow$ means row operation

$R_3 -= 3R_1$ $\rightsquigarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ \boxed{\begin{aligned} -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned}} \end{aligned}$$

We've eliminated x_1 from eqs. 2 & 3!

$R_3 -= \frac{5}{7}R_2$ $\rightsquigarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 0 \boxed{- \tfrac{50}{7}x_3 = -\tfrac{150}{7}} \end{aligned}$$

Now easy to solve for x_3,

$x_3 = 3$

$-7x_2 - 4 \cdot 3 = 2, \quad x_2 = -2$

$x_1 + 2(-2) + 3(3) = 6$

$x_1 - 4 + 9 = 6$, $x_1 = 1$

Solution is $x = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$

Check $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ satisfies original system

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

Aside:

$Ax = b$

$$\begin{aligned} 1 - 4 + 9 &= 6 \\ 2 + 6 + 6 &= 14 \\ 3 - 2 - 3 &= -2 \end{aligned}$$

Example:

$R_1 \quad 4x_2 + 3x_3 = 2$

$R_2 \quad x_1 + x_2 - x_3 = 3$

swap

$R_3 \quad 2x_1 - 3x_2 - 6x_3 = -3$

x_1 is already eliminated from R_1,

so swap with R_2.

swap

$R_1 \leftrightarrow R_2$ $\rightsquigarrow$

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned}$$

$R_3 -= 2R_1$ $\rightsquigarrow$

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 0 \quad -5x_2 - 4x_3 &= -9 \end{aligned}$$

$R_3 += \frac{5}{4}R_2$ $\rightsquigarrow$

$$\begin{aligned} x_1 + x_2 - x_3 &= 3 \\ 4x_2 + 3x_3 &= 2 \\ 0 \quad -\tfrac{1}{4}x_3 &= -\tfrac{13}{2} \end{aligned}$$

Solve via backsubstitution:

$x_3 = 26$

$4x_2 + 3(26) = 2, \quad x_2 = -19$

$x_1 - 19 - 26 = 3, \quad x_1 = 48$

So $x = \begin{pmatrix} 48 \\ -19 \\ 26 \end{pmatrix}$

Example:

$$\boxed{\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= -1 \end{aligned}}$$

$R_2 -= 4R_1$ $\rightsquigarrow$

$R_3 -= 7R_1$ $\rightsquigarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ \boxed{\begin{aligned} -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -8 \end{aligned}} \end{aligned}$$

$R_3 -= 2R_2$ $\rightsquigarrow$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We're done!

Can choose any value for x_3 and solve for x_1, x_2:

$$x_2 = \frac{4}{3} - 2x_3$$

$$x_1 = 1 - 3x_3 - 2x_2 = -\frac{5}{3} + x_3$$

Solutions: $\begin{pmatrix} -5/3 + t \\ 4/3 - 2t \\ t \end{pmatrix}, \ t \in \mathbb{R}$

for instance, $\begin{pmatrix} -2/3 \\ -2/3 \\ 1 \end{pmatrix}$ when $t=1$

So there are infinitely many sol. (intersection of 3 planes, two of which are identical)

Example: $x_1 + 2x_2 + 3x_3 = 1$

$4x_1 + 5x_2 + 6x_3 = 0$

$7x_1 + 8x_2 + 9x_3 = 0$ → only change from prev. ex.

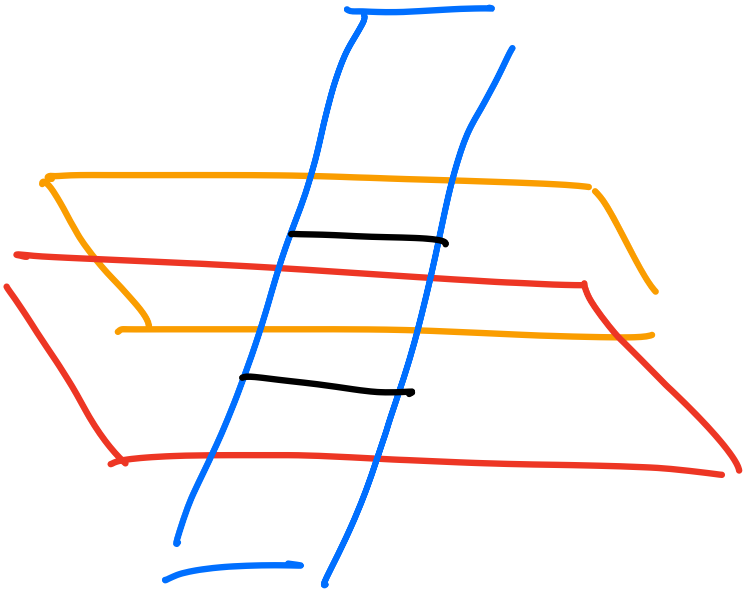
$R_2 - = 4R_1 \leadsto$, $R_3 - = 7R_1$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 0 - 3x_2 - 6x_3 &= -4 \\ 0 - 6x_2 - 12x_3 &= -7 \end{aligned}$$

$R_3 - = 2R_2 \leadsto$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 1 \end{aligned}$$

If our original equations were solvable then $0=1$, not true, so original system not solvable.



Row Operations

(1) Row Replacement

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \qquad R_2 -= 2R_1 \rightsquigarrow$$

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 0 - 7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

Replaced R_2 by $R_2 - 2R_1$

(2) Row Swap

(3) Scalar Multiplication by
⚠ nonzero scalar

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$
$$3x_1 + x_2 - x_3 = -2$$

$R_1 \times= 2$ $\rightsquigarrow$

$$2x_1 + 4x_2 + 6x_3 = 12$$
$$2x_1 - 3x_2 + 2x_3 = 14$$
$$3x_1 + x_2 - x_3 = -2$$

Fact: All these operations are reversible and do not change set of solutions.

Only non-obvious one is row replacement:

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = a_0$$
$$b_1x_1 + b_2x_2 + \dots + b_nx_n = b_0$$

$R_2 \mathrel{+}= cR_1$ $\quad$ $R_2 \mathrel{-}= cR_1$

$$a_1x_1 + \dots + a_nx_n = a_0$$

$$(b_1+ca_1)x_1 + \dots + (b_n+ca_n)x_n = b_0+ca_0$$

Three Ways of Writing System of Lin Eqs.

① System of eqs.

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$

② Matrix equation

$$\underset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 6 \\ 14 \end{bmatrix}}$$

the <u>coefficient</u> <u>matrix</u> A simply comes from the coeff. of the lin. system and if A has m rows and n columns then the system has m equations and n variables.

③ Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] = \left[A \mid b \right]$$

Augmented matrices are nice for row operations b/c row operations only change coefficients.

<u>Example:</u>

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \quad \begin{array}{l} R_2 -= 2R_1 \\ \rightsquigarrow \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

What does it mean to be done with the algorithm?

Def: A matrix is in row echelon form (REF) if

1. The first nonzero entry of each row is to the right of the first nonzero entry of the row above
2. All zero rows at the bottom.

✓ $\begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$

$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix}$ ✗ $\begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ✗

⚠ When checking if an augmented matrix is in REF, ignore augmentation line.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right], \quad \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \checkmark$$

Solving a linear system

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Putting the augmented matrix into REF using the 3 row operations.

Def: The pivot positions of a matrix are the 1st nonzero entries of each row when the matrix is in REF

Well defined?

The rank of a matrix is the # of pivot positions.