1.13.22

Tuesday in Math&Physics Room 130!

Number of Solutions

<u>Recall</u>: The <u>pivots</u> of a REF matrix are the positions of the first nonzero entries in each row.

⚠ The pivots of any matrix are the pivots of any of its REF forms.

The <u>rank</u> of a matrix is the # of pivots.

<u>Example</u>:

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$
$$3x_1 - x_2 - x_3 = -2$$

$$x_1 + 2x_2 + 3x_3 = 6$$
$$-7x_2 - 4x_3 = 2$$
$$-\frac{50}{7}x_3 = \frac{-150}{7}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & -1 & -1 & -2 \end{array}\right] \xrightarrow{\text{REF}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

Here we have exactly one solution. Found via back-substitution.

Example:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{array}\right] \overset{\text{REF}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Have ∞ solutions b/c we can choose any value for x_3.

Example:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] \overset{\text{REF}}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

No solutions.

Def: A pivot column of a matrix is a column with a pivot.

Procedure for determining # of solutions:

① Put the matrix into REF using row ops.

② Locate pivots

③ Determine pivot columns.

← Pivot column

Not pivot col.

Fact: Let A be an augmented matrix corresponding to a linear system of equations.

① If every column except the last is a pivot column then the system has one solution.

∞ If the last column and some other column are not pivot columns there are ∞ many solutions.

⓪ If the last column has a pivot there are no solutions.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & -2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$ has 1 solution.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$ has ∞ solutions

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right]$$ no solutions.

Def: A system is *consistent* if it has at least one solution; *inconsistent* otherwise.

Gaussian Elimination (GE)

⚠ Almost all problems in this course reduce to GE

Def: Two matrices are *row equivalent* if you can get from one to the other using only row ops.

Note: If two aug. matrices are row eq. then they have the same solution set.

? (Fun) If two aug. matrices have the same sol. set are they row eq.?

<u>Algo</u> (GE)

Input: any matrix

Outputs: A row eq. matrix in REF.

Procedure:

(1a) If the first nonzero column has a zero entry at the top, swap it with some row so that top entry nonzero.

$$\begin{bmatrix} 0 & 4 & 3 & 2 \\ 1 & 1 & -1 & 3 \\ 2 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 2 \\ 2 & -3 & -6 & -3 \end{bmatrix}$$

This is now the first pivot position.

(1b) Perform row replacements using R_1 to clear entries below first pivot

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 2 \\ 2 & -3 & -6 & -3 \end{bmatrix} \xrightarrow{R_3 -= 2R_1} \begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & 3 \end{bmatrix}$$

Now recurse on submatrix below and to the right of pivot.

Now apply steps (1a) and (1b) to submatrix.

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 2 \\ 0 & -5 & -4 & 3 \end{bmatrix}$$

(1a) doesn't apply

$\big\downarrow R_3 \mathrel{+}= \frac{5}{4} R_2$

$$\begin{bmatrix} 1 & 1 & -1 & 3 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & -1/4 & 11/2 \end{bmatrix}$$

<u>Ex:</u>

$$\boxed{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 3 \\ 0 & 0 & 1 & 1 \end{bmatrix}} \xrightarrow[R_2 \mathrel{-}= 2R_1]{(1b)}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

First nonzero column!

(2b)

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

⚠ If you want to apply to augmented matrix, ignore aug. line.

Jordan Substitution:

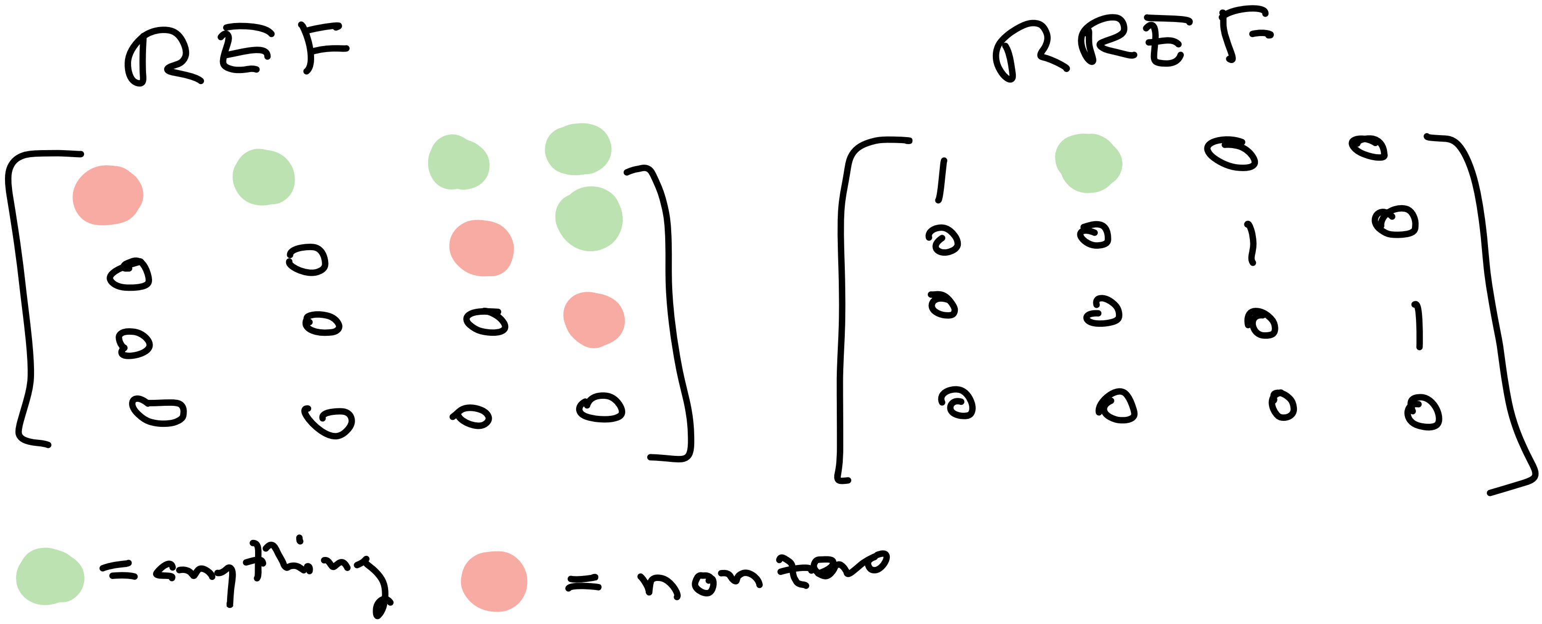
This is the algo for back-substitution.
It puts matrix in following form:

Def: A matrix is in Reduced Row Echelon Form (RREF) if

(1-2) It's in REF

(3) All pivots are 1.

(4) A pivot is the only nonzero entry in its column.



Algo (Jordan Substitution)

Input: REF matrix

Output: The row-equivalent matrix in RREF

Procedure:

Loop, starting at *last* pivot.

(a) Scale pivot row so pivot $= 1$

(b) Use row replacements to clear column above that pivot.

Thm: The RREF of a matrix is unique.

Gaussian Elim. + Jordan Substitution
(Gauss-Jordan Algorithm)

converts any matrix to its unique RREF form.

<u>Example:</u> Jordan Sub.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right] \overset{(a)}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\underset{\substack{R_2 += 4R_3 \\ R_1 -= 3R_3}}{\overset{(b)}{\rightsquigarrow}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & -7 & 0 & 14 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Done with first loop

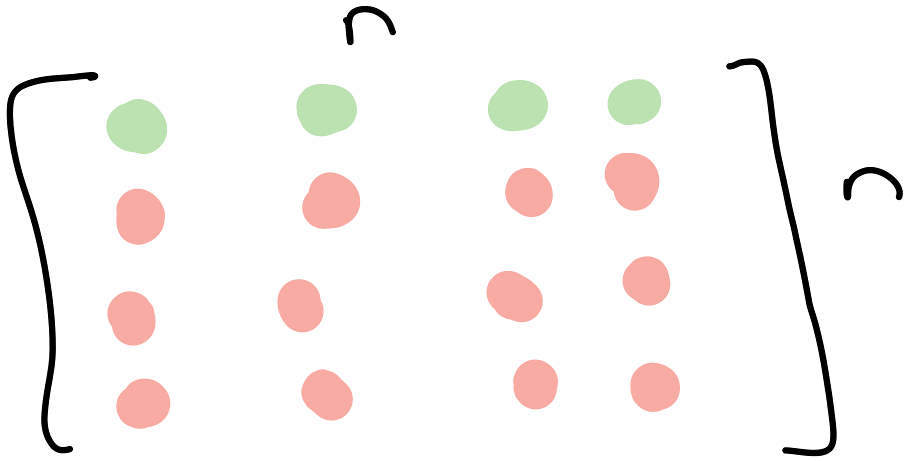
$$\underset{R_2 \times= -\frac{1}{7}}{\overset{(a)}{\rightsquigarrow}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\underset{R_1 -= 2R_2}{\overset{(b)}{\rightsquigarrow}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

eg: lin. system. $\begin{cases} x_1 = 1 \\ x_2 = -2 \\ x_3 = 3 \end{cases}$

Computational Complexity.

Gaussian Elimination.



Total flops (floating point operations)

$$2 \cdot \frac{n(n-1) \cdot\cdot (2n-1)}{6} \approx \frac{2}{3} n^3$$

Jordan substitution

takes n^2

Note: $\frac{2}{3} n^3$ much bigger than n^2