Geometry of Diagonalizable Matrices

Last time: an $n\times n$ matrix A is diagonalizable if it has an eigenbasis $\{w_1, \ldots, w_n\}$.

Matrix form: $A = CDC^{-1}$ where

$$C = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

In this case every vector can be written as a linear combination of eigenvectors, so the action of A on $\mathbb{R}^n$ is reduced to scalar multiplication:

$$A(x_1 w_1 + \cdots + x_n w_n) = \lambda_1 x_1 w_1 + \cdots + \lambda_n x_n w_n.$$

What does this mean geometrically?

→ Expanding in an eigenbasis and scalar multiplication can both be formulated geometrically!

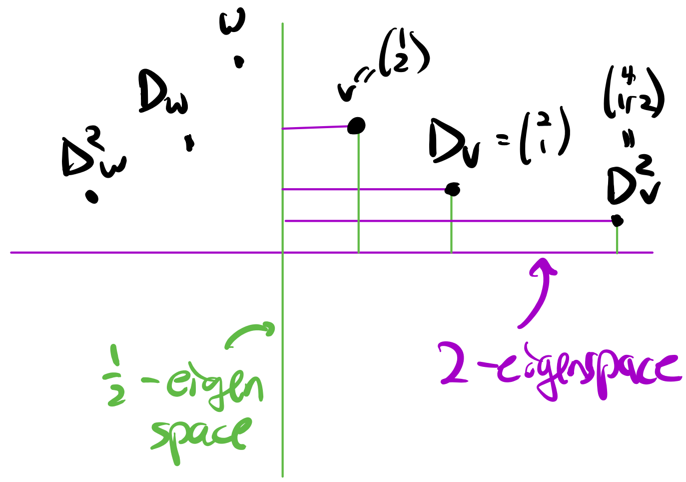
NB: "Visualizing" a matrix means understanding how x relates to Ax: think of A as a function

$$\underset{\text{input}}{x} \rightsquigarrow \underset{\text{output}}{Ax}$$

Eg: $D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$ so $D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ \frac{1}{2}y \end{pmatrix}$

- scales the x-direction by 2
- scales the y-direction by $1/2$

[demo]



Eg: $A = \frac{1}{10}\begin{pmatrix} 11 & 6 \\ 9 & 14 \end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{5}{2}\lambda + 1 = (\lambda - 2)(\lambda - \frac{1}{2})$

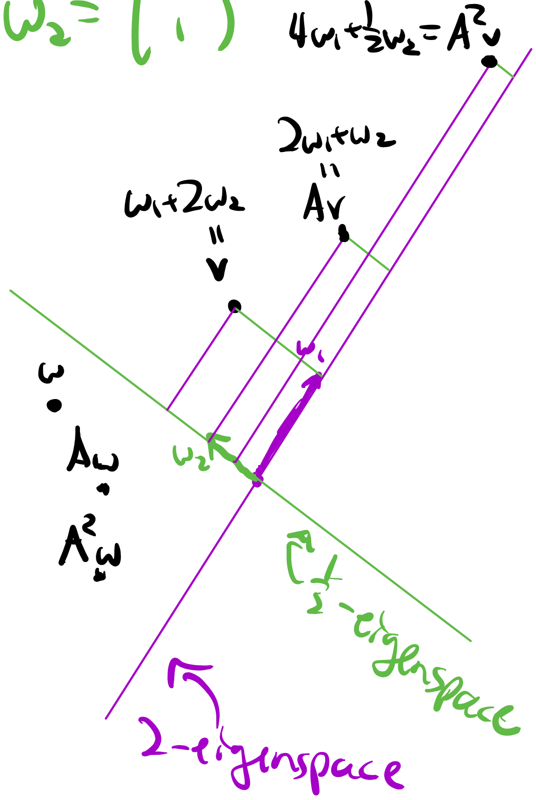
$\lambda_1 = 2 \quad w_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Expand in the eigenbasis!
(think in terms of LCs of w_1, w_2)

$$A(x_1 w_1 + x_2 w_2) = 2x_1 w_1 + \frac{1}{2} x_2 w_2$$

- scales the w_1-direction by 2
- scales the w_2-direction by $1/2$

[demo]



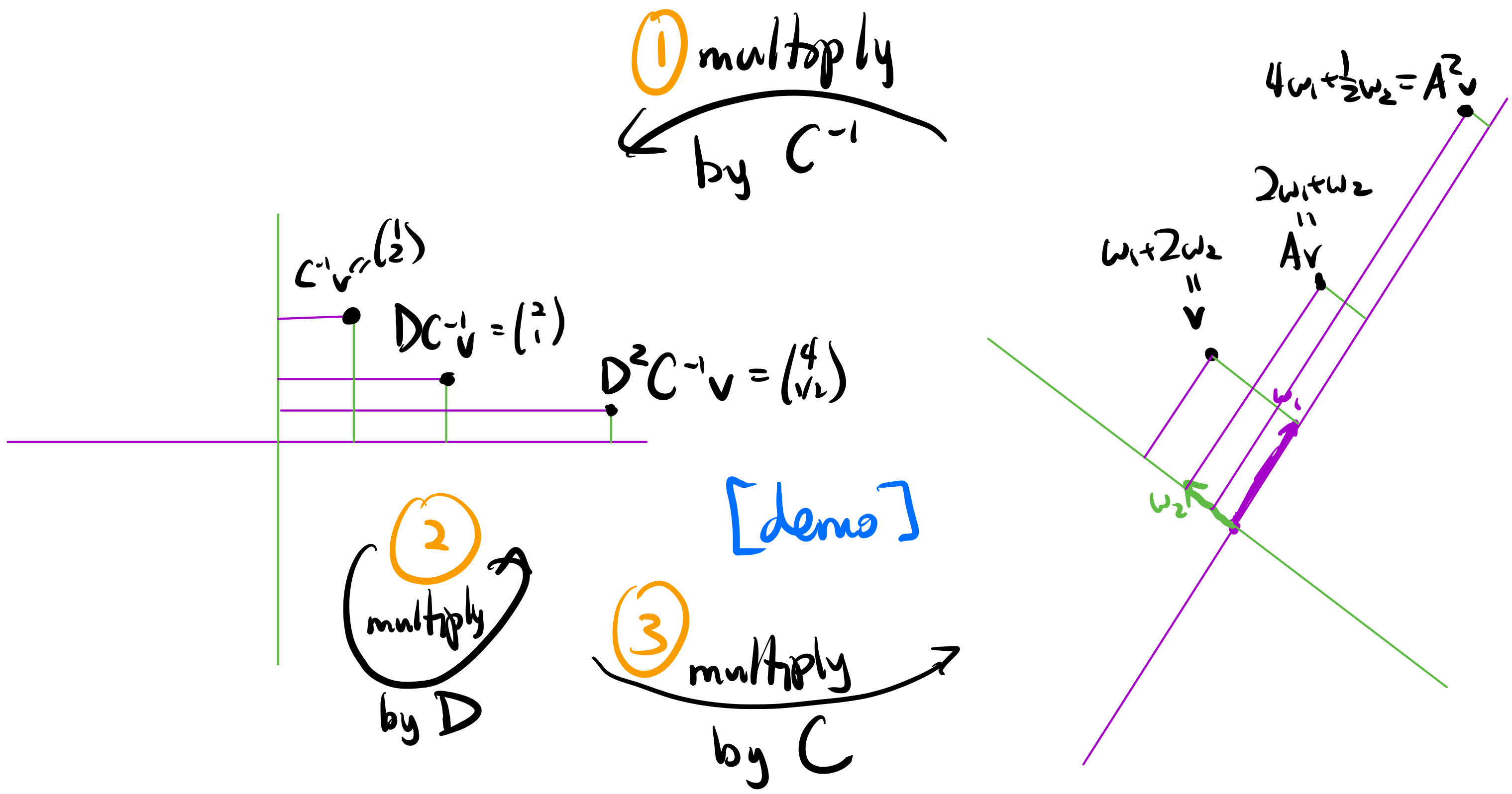
This is the vector form. In matrix form

$$A = CDC^{-1} \qquad C = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Then $Av = CDC^{-1}v$

= " • first multiply v by C^{-1}
• then multiply by the diagonal matrix D
• then multiply by C again"

Note $C\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 w_1 + x_2 w_2 \iff C^{-1}(x_1 w_1 + x_2 w_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

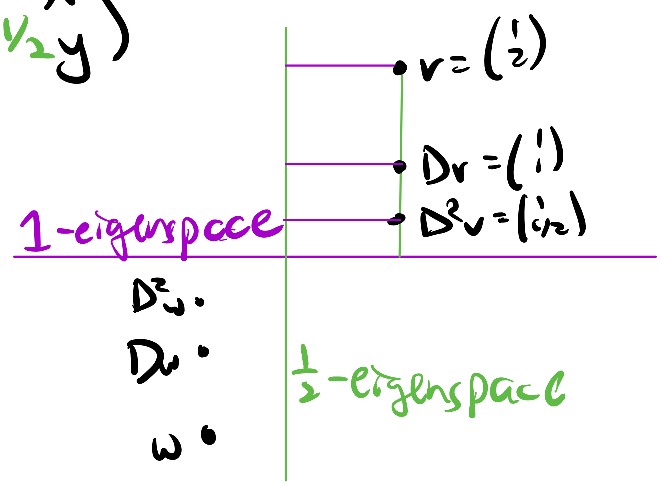


[demo]

[More examples we didn't do in class:]

Eg: $D = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$ $\quad D\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1x \\ \frac{1}{2}y \end{pmatrix}$

- scales the x-direction by 1
- scales the y-direction by $\frac{1}{2}$



[demo]

Eg: $A = \frac{1}{6}\begin{pmatrix}5 & 1\\ 2 & 4\end{pmatrix}$ $\quad p(\lambda) = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda-1)(\lambda-\frac{1}{2})$

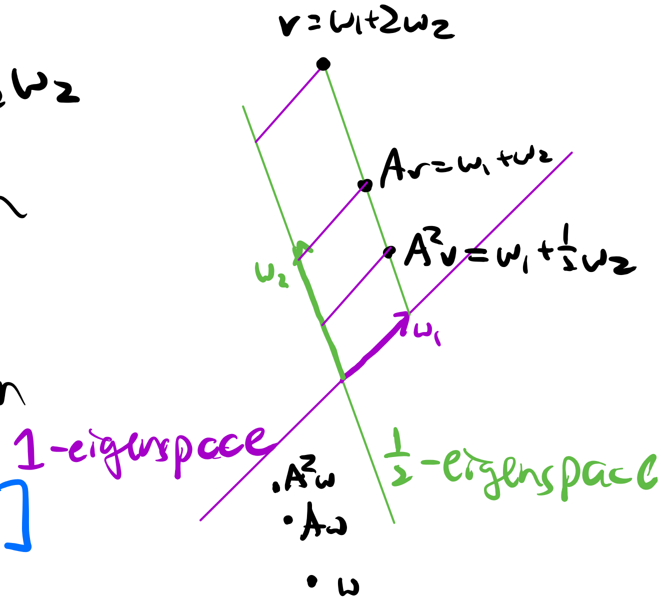
$\lambda_1 = 1 \quad w_1 = \begin{pmatrix}1\\1\end{pmatrix} \qquad \lambda_2 = \frac{1}{2} \quad w_2 = \begin{pmatrix}-1\\2\end{pmatrix}$

Expand in the eigenbasis!

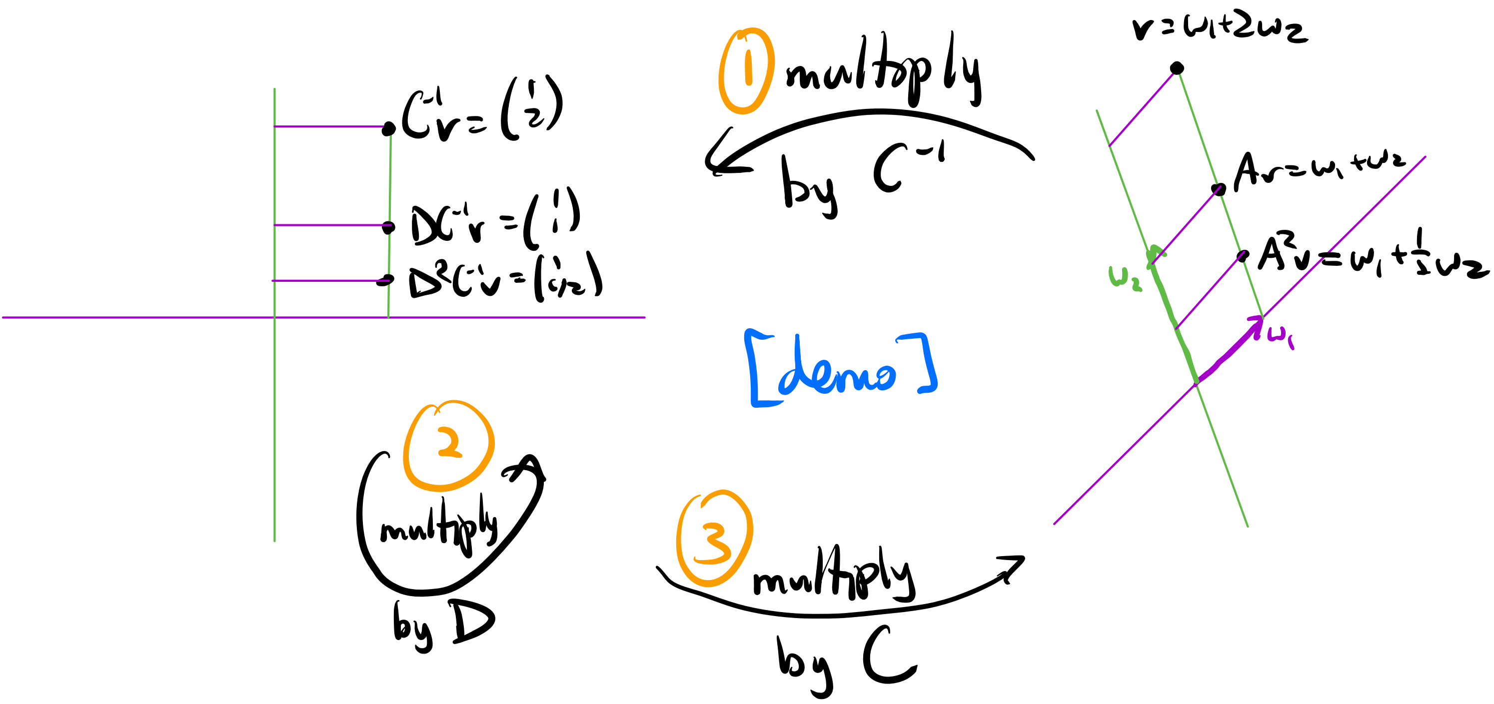
$$A(x_1w_1 + x_2w_2) = 1x_1w_1 + \frac{1}{2}x_2w_2$$

- scales the w_1-direction by 1
- scales the w_2-direction by $1/2$

[demo]



Matrix Form: $A = CDC^{-1}$ $\quad C = \begin{pmatrix}1 & -1\\ 1 & 2\end{pmatrix}$ $\quad D = \begin{pmatrix}1 & 0\\ 0 & 1/2\end{pmatrix}$



Eg: $A = \frac{1}{580}\begin{pmatrix} 503 & 73 & 269 \\ 207 & 1137 & -49 \\ 270 & -30 & 680 \end{pmatrix}$ has eigenbasis

$$w_1 = \begin{pmatrix} -7 \\ 2 \\ 5 \end{pmatrix} \qquad w_2 = \begin{pmatrix} -1 \\ -9 \\ 0 \end{pmatrix} \qquad w_3 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$$

and eigenvalues

$$\lambda_1 = 1/2 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3/2$$

Expand in the eigenbasis!

$$A(x_1w_1 + x_2w_2 + x_3w_3) = \frac{1}{2}x_1w_1 + 2x_2w_2 + \frac{3}{2}x_3w_3$$

- scales the w_1-direction by $\frac{1}{2}$
- scales the w_2-direction by 2
- scales the w_3-direction by $\frac{3}{2}$

[demo]

Diagonalization with Complex Eigenvalues

Diagonalization still works great even if the eigenvalues are not real.

→ Still can solve difference equations & ODEs

→ Still get real-number answers.

Fact: The complex eigenvalues & eigenvectors of a real matrix come in complex conjugate pairs:

$$Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$$

here $v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \rightsquigarrow \bar{v} = \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}$

Eg: Solve the difference equation

$v_{k+1} = A v_k \quad A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \quad v_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ || No complex #s in the statement!

(1) Diagonalize:

$$p(\lambda) = \lambda^2 + 3\lambda + 3 \Rightarrow \lambda = \tfrac{1}{2}\left(-3 \pm \sqrt{9-12}\right)$$

$$\rightsquigarrow \lambda = \tfrac{1}{2}(-3 + i\sqrt{3}), \quad \bar{\lambda} = \tfrac{1}{2}(-3 - i\sqrt{3})$$

Find eigenvectors using the 2×2 trick (HW10)

$$w = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \qquad \bar{w} = \begin{pmatrix} 1 \\ -\bar{\lambda} \end{pmatrix}$$

eigenvector for λ eigenvector for $\bar{\lambda}$

Check: $Aw = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$

Wait, is this equal to λw?

$$\lambda w = \lambda \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda^2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$$

Yes: $-\lambda^2 = 3+3\lambda$ because

$\lambda^2+3\lambda+3 = p(\lambda) = 0$ ✓

So $\{w, \bar{w}\}$ is an eigenbasis

(different eigenvalues $\Rightarrow$ LI)

(2) Expand the initial state in our eigenbasis:

We need to solve $\begin{pmatrix} 2 \\ 3 \end{pmatrix} = v_0 = x_1 w + x_2 \bar{w}$.

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -\lambda & -\bar{\lambda} & 3 \end{array}\right) \xrightarrow{R_2 += \lambda R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & \lambda-\bar{\lambda} & 3+2\lambda \end{array}\right)$$

$3+2\lambda = i\sqrt{3}$

$\lambda-\bar{\lambda} = 2i\,\mathrm{Im}(\lambda) = i\sqrt{3}$

$$\rightsquigarrow \begin{cases} x_1 + x_2 = 2 \\ (i\sqrt{3})x_2 = i\sqrt{3} \end{cases} \rightsquigarrow x_2 = 1 \longrightarrow x_1 = 1$$

(back-substitution)

So $v_0 = w_1 + w_2$

Solution: $A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w}$

So far it's exactly the same as for real eigenvalues!

... but we wanted a solution involving only real #s.

Thankfully, $\lambda^k w$ and $\bar{\lambda}^k \bar{w}$ are complex conjugates, so

$$A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w} = 2\,\mathrm{Re}[\lambda^k w]$$

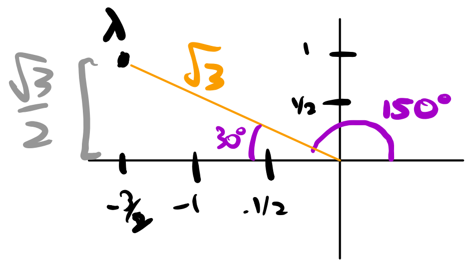
$$= 2\,\mathrm{Re}\left[\lambda^k \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}\right] = 2\,\mathrm{Re}\begin{pmatrix} \lambda^k \\ -\lambda^{k+1} \end{pmatrix}$$

Recall: Multiplication of complex numbers is much easier in polar form.

$$\lambda = \frac{1}{2}(-3 + i\sqrt{3}) = r e^{i\theta}$$

$$r = \frac{1}{2}\sqrt{9+3} = \frac{1}{2}\sqrt{4\cdot 3} = \sqrt{3}$$

$$\theta = 150^\circ = 5\pi/6$$



Euler's Formula

So $$\lambda^k = r^k e^{ik\cdot\frac{5\pi}{6}} = (\sqrt{3})^k\left(\cos\frac{5k\pi}{6} + i\sin\frac{5k\pi}{6}\right)$$

$$\Rightarrow \mathrm{Re}(\lambda^k) = (\sqrt{3})^k \cos\frac{5k\pi}{6}$$

$$\Rightarrow \boxed{v_k = 2\begin{pmatrix} \sqrt{3}^k \cos(5k\pi/6) \\ -\sqrt{3}^{k+1}\cos(5(k+1)\pi/6) \end{pmatrix}}$$

[demo]

The answer involves only real numbers (and cosines - weird!) but we needed complex numbers to get it!

Difference Equations with Complex Eigenvalues:

To solve $v_{k+1} = Av_k$:

(1-2) Diagonalize A and expand v_0 in an eigenbasis, as before. Complex numbers are OK.

$\rightarrow$ Remember $Av = \lambda v \Leftrightarrow A\bar{v} = \bar{\lambda}\bar{v}$

(3) Group complex conjugate terms:

$$\lambda^k x\omega + \bar{\lambda}^k \bar{x}\bar{\omega} = 2\,\mathrm{Re}(\lambda^k x\omega)$$

(4) Write λ in polar form:

$$\lambda = re^{i\theta} \Rightarrow \lambda^k = r^k e^{ik\theta} = r^k(\cos k\theta + i\sin k\theta)$$

Multiply this by x and the coordinates of ω and take the real part

$\rightsquigarrow$ get an answer with sines & cosines (but no i's).

Algebraic & Geometric Multiplicity

Last, we will discuss a criterion for diagonalizability.

(We like diagonalizable matrices because we can solve difference equations)

Recall: If λ is a root of a polynomial $p(x)$, its multiplicity m is the largest power of $(x-\lambda)$ dividing p:

$$p(x) = (x-\lambda)^m \cdot (\text{other factors})$$

Def: Let A be an $n \times n$ matrix with eigenvalue λ.

(1) The algebraic multiplicity (AM) of λ is its multiplicity as a root of the characteristic polynomial $p(\lambda)$.

(2) The geometric multiplicity (GM) of λ is the dimension of the λ-eigenspace:

$$\begin{aligned} GM(\lambda) &= \dim \operatorname{Nul}(A - \lambda I_n) \\ &= \#\text{free variables in } A - \lambda I_n. \\ &= \#\text{linearly independent } \lambda\text{-eigenvectors} \end{aligned}$$

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

So the eigenvalues are 1 & 2.

- $\lambda = 1$: AM $= 1$.

  $\mathrm{Nul}(A - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ AM = GM ✓

  → this is a line: GM $= 1$

- $\lambda = 2$: AM $= 2$

  $\mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\}$ AM ≠ GM ✓

  → this is a line: GM $= 1$

This matrix is not diagonalizable:

only two linearly independent eigenvectors.

[demo]

Eg: $B = \begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

So the eigenvalues are 1 & 2.

- $\lambda = 1$: AM $= 1$

  $\mathrm{Nul}(B - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ AM = GM ✓

  → this is a line: GM $= 1$

- $\lambda = 2$: AM $= 2$

  $\mathrm{Nul}(B - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$ AM = GM ✓

  → this is a plane: GM $= 2$

This matrix is diagonalizable: an eigenbasis is

$$\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\4\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$$

[demo]

Both matrices have only 2 eigenvalues.

The difference is that B had AM=GM=2 LI 2 eigenvectors and A had one.

Thm (AM ≥ GM): For any eigenvalue λ of A,

(algebraic multiplicity of λ)

$\geq$ (geometric multiplicity of λ) ≥ 1

NB: GM ≥ 1 just says every eigenvalue has an eigenvector — the eigenspace can't be $\{0\}$, so its dimension is ≥ 1.

NB: This is one of the few facts I can't prove for you in the notes.

Upshot: if $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$ then

- the 1-eigenspace is necessarily a line:
  $$AM = 1 \geq GM \geq 1$$
- the 2-eigenspace is a line or a plane:
  $$AM = 2 \geq GM \geq 1$$
- the matrix is diagonalizable $\iff GM(2) = 2$: then you have $1+2=3$ LI eigenvectors.

Thm (AM/GM Criterion for Diagonalizability):

Let A be an $n\times n$ matrix.

- A is diagonalizable over the complex numbers
  $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ
- A is diagonalizable over the real numbers
  $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ
  and A has no complex eigenvalues.

Eg: $B = \begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ is not diagonalizable because
$$AM(2) = 2 \neq 1 = GM(2)$$

Corollary: If A has n different eigenvalues then A is diagonalizable.

Proof: If A has n different eigenvalues then

$$n = AM(\lambda_1) + \cdots + AM(\lambda_n) \Longrightarrow AM(\lambda_i) = 1$$
$$1 = AM(\lambda_i) \geq GM(\lambda_i) \geq 1 \Longrightarrow AM(\lambda_i) = GM(\lambda_i) = 1$$

Eg: A 2×2 real matrix with a complex eigenvalue λ is diagonalizable (over $\mathbb{C}$): it has 2 eigenvalues λ and $\bar{\lambda}$.

Proof of the Theorem:

First note that

$$p(\lambda) = (-1)^n(\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_r)^{m_r}$$

factors into linear factors (over $\mathbb{C}$), where $m_i = AM(\lambda_i)$. Hence

$$\boxed{AM(\lambda_1) + \cdots + AM(\lambda_r) = n \qquad \text{(sum of the AM's is } n\text{)}}$$

If A is diagonalizable then it has n LI eigenvectors, so

$$\begin{array}{ccccc} n = & GM(\lambda_1) & + \cdots + & GM(\lambda_n) & \\ & \wedge\!| & & \wedge\!| & \\ & AM(\lambda_1) & + \cdots + & AM(\lambda_n) & = n \end{array}$$

This forces $AM(\lambda_i) = GM(\lambda_i)$. Conversely, if each $AM(\lambda_i) = GM(\lambda_i)$ then

$$n = GM(\lambda_1) + \cdots + GM(\lambda_n),$$

so when you combine eigenspace bases you get n LI eigenvectors.

✓