Solving Systems of Equations using Elimination

Here's a system of 3 equations in 3 variables:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases}$$

How to solve it?

- **Substitution:** solve 1st equation for x_1, substitute into 2nd & 3rd, continue.
- **Elimination:** "combine" the equations to eliminate variables.

Elimination turns out to scale much better (to more equations & variables), so we'll focus on that.

"replace the 2nd equation with the 2nd minus 2× the 1st"

Eg:

$$\begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{array} \xrightarrow{R_2 -= 2R_1} \begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -7x_2 - 4x_3 = 2 \\ 3x_1 + x_2 - x_3 = -2 \end{array}$$

$$\xrightarrow{R_3 -= 3R_1} \begin{array}{l} x_1 + 2x_2 + 3x_3 = 6 \\ \quad -7x_2 - 4x_3 = 2 \\ \quad -5x_2 - 10x_3 = -20 \end{array}$$

Now we have eliminated x_1 from the 2nd & 3rd eq.s

These now form 2 equations in 2 variables: simpler!

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -5x_2 - 10x_3 &= -20 \end{aligned} \quad \xrightarrow{R_3 \to -\frac{5}{7}R_2} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ -\frac{50}{7}x_3 &= -\frac{150}{7} \end{aligned}$$

We eliminated x_2 from the last equation: now it's one equation in one variable. Easy!

We can now solve via back-substitution:

$$-\frac{50}{7}x_3 = -\frac{150}{7} \implies x_3 = 3.$$

Substitute into 2nd equation:

$$-7x_2 - 4x_3 = 2 \rightsquigarrow -7x_2 - 4 \cdot 3 = 2$$

Now solve for x_2:

$$-7x_2 - 12 = 2 \implies -7x_2 = 14 \implies x_2 = -2$$

Substitute both into 1st equation:

$$x_1 + 2x_2 + 3x_3 = 6 \rightsquigarrow x_1 + 2 \cdot (-2) + 3 \cdot 3 = 6$$

Now solve for x_1:

$$x_1 - 4 + 9 = 6 \implies x_1 = 1$$

Check:

$$\begin{aligned} 1 + 2(-2) + 3(3) &= 6 \\ 2 \cdot 1 - 3(-2) + 2(3) &= 14 \\ 3 \cdot 1 + (-2) - 3 &= -2 \end{aligned} \quad \checkmark$$

Does this always work?

Eg:

$$4x_2 + 3x_3 = 2$$
$$x_1 + x_2 - x_3 = 3$$
$$2x_1 - 3x_2 - 6x_3 = -3$$

(swap)

x_1 is already eliminated from R_1. Fix: swap the 1st 2 eqns.

$\xrightarrow{R_1 \leftrightarrow R_2}$

$$x_1 + x_2 - x_3 = 3$$
$$4x_2 + 3x_3 = 2$$
$$2x_1 - 3x_2 - 6x_3 = -3$$

Now eliminate as before:

$\xrightarrow{R_3 -= 2R_1}$

$$x_1 + x_2 - x_3 = 3$$
$$4x_2 + 3x_3 = 2$$
$$-5x_2 - 4x_3 = -9$$

$\xrightarrow{R_3 += \frac{5}{4}R_2}$

$$x_1 + x_2 - x_3 = 3$$
$$4x_2 + 3x_3 = 2$$
$$-\frac{1}{4}x_3 = -\frac{13}{2}$$

Solve using back-substitution:

$$-\frac{1}{4}x_3 = -\frac{13}{2} \implies x_3 = 26$$

Substitute into 2nd equation:

$$4x_2 + 3(26) = 2 \implies x_2 = -19$$

Substitute both into 1st equation:

$$x_1 - 19 - 26 = 3 \implies x_1 = 48$$

Check:

$$\begin{aligned} 4x_2 + 3x_3 &= 2 \\ x_1 + x_2 - x_3 &= 3 \\ 2x_1 - 3x_2 - 6x_3 &= -3 \end{aligned} \rightsquigarrow \begin{aligned} 4(-19) + 3(26) &= 2 \\ 48 - 19 - 26 &= 3 \\ 2(48) - 3(-19) - 6(26) &= -3 \end{aligned}$$

✓

Eg:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= -1 \end{aligned} \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -8 \end{aligned}$$

$$\xrightarrow{R_3 -= 2R_2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 0 \end{aligned}$$

Are we done? Yes: choose any value for x_3, then back-substitute to find x_1, x_2:

$$-3x_2 = -4 + 6x_3 \implies x_2 = \frac{4}{3} - 2x_3$$

$$x_1 = 1 - 2x_2 - 3x_3 = 1 - \frac{8}{3} + 4x_3 - 3x_3$$

$$x_1 = -5/3 + x_3$$

Eg: $x_3 = 1 \rightsquigarrow x_1 = -2/3,\ x_2 = -2/3$

Check:

$$\begin{aligned} -2/3 - 4/3 + 3 &= 1 \\ -8/3 - 10/3 + 6 &= 0 \\ -14/3 - 16/3 + 9 &= -1 \end{aligned}$$

✓

In this case there are infinitely many solutions. We'll deal with this in Week 4.

Eg: $x_1 + 2x_2 + 3x_3 = 1$, $4x_1 + 5x_2 + 6x_3 = 0$, $7x_1 + 8x_2 + 9x_3 = 0$ (tweak previous example)

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 4x_1 + 5x_2 + 6x_3 &= 0 \\ 7x_1 + 8x_2 + 9x_3 &= 0 \end{aligned} \xrightarrow[R_3 -= 7R_1]{R_2 -= 4R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ -6x_2 - 12x_3 &= -7 \end{aligned}$$

$$\xrightarrow{R_3 -= 2R_2} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ -3x_2 - 6x_3 &= -4 \\ 0 &= 1 \end{aligned}$$

If our original equations were true, then $0=1$.
Thus our system has no solutions.
(last 2 eqns are parallel planes)

Row Operations are the allowed manipulations we can perform on our equations.

(1)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_2 -= 2R_1} \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row replacement
replace R_2 by $R_2 - 2R_1$

(2)
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned} \xrightarrow{R_1 \leftrightarrow R_2} \begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 14 \\ x_1 + 2x_2 + 3x_3 &= 6 \\ 3x_1 + x_2 - x_3 &= -2 \end{aligned}$$

row swap
(change order)

(3) $x_1 + 2x_2 + 3x_3 = 6$ $\quad R_1 \times = 2 \quad$ $2x_1 + 4x_2 + 6x_3 = 12$

$2x_1 - 3x_2 + 2x_3 = 14 \quad \rightsquigarrow \quad 2x_1 - 3x_2 + 2x_3 = 14$

$3x_1 + x_2 - x_3 = -2 \qquad\qquad 3x_1 + x_2 - x_3 = -2$

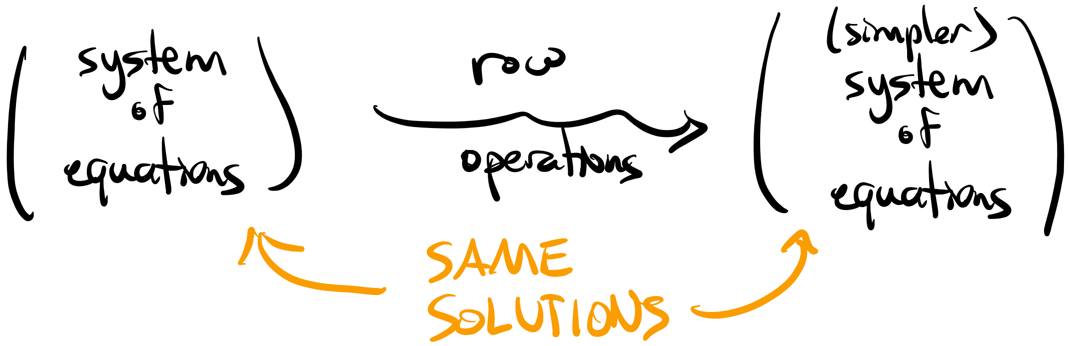
scalar multiplication
(by nonzero scalar)

Obviously if (x_1, x_2, x_3) is a solution before doing a row operation, then it is true after. Eg. row replacement:

$x_1 + 2x_2 + 3x_3 \to 6 = 6$ $\quad R_2 -= 2R_1 \quad$ $x_1 + 2x_2 + 3x_3 \to 6 = 6$

$2x_1 - 3x_2 + 2x_3 \to 14 = 14 \quad \rightsquigarrow \quad -7x_2 - 4x_3 \to 2 = 2.$

Fact: All these operations are reversible: if you have a solution (x_1, x_2, x_3) after doing a row operation, then it's also a solution before.



This was the whole point: we wanted to solve our (original) system of equations!

Questions: How do you undo (reverse):

- $R_1 += R_2$? $\quad R_1 -= R_2$
- $R_1 \times = 2$? $\quad R_1 \div = 2$
- $R_1 \leftrightarrow R_2$? $\quad R_1 \leftrightarrow R_2$

The variables $x_1, x_2, \ldots$ are just placeholders; only their coefficients matter. Let's extract them into a matrix.

Three Ways to Write System of Linear Equations

(1) As a system of equations:

$$x_1 + 2x_2 + 3x_3 = 6$$
$$2x_1 - 3x_2 + 2x_3 = 14$$

(2) As a matrix equation $Ax = b$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 6 \\ 14 \end{bmatrix}}_{b}$$

If you expand out the product you get

$$\begin{bmatrix} x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

which is what we had before.

The coefficient matrix A comes from the coefficients of the variables:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \end{bmatrix} \longleftrightarrow \begin{matrix} 1x_1 + 2x_2 + 3x_3 \\ 2x_1 - 3x_2 + 2x_3 \end{matrix}$$

The vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ contains the unknowns or variables.

NB: A is an $m \times n$ matrix where

m = # equations $\qquad b \in \mathbb{R}^m$ ← size m

n = # variables $\qquad x \in \mathbb{R}^n$ ← size n

(3) As an augmented matrix.

This is a notational convenience: just squash A & b together and separate with a line.

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \longleftrightarrow \begin{aligned} 1x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \end{aligned}$$

$$\| \qquad [A \mid b]$$

Augmented matrices are good for row operations, which only affect the coefficients (not the variables):

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ 2x_1 - 3x_2 + 2x_3 &= 14 \end{aligned} \quad \xrightarrow{R_2 -= 2R_1} \quad \begin{aligned} x_1 + 2x_2 + 3x_3 &= 6 \\ -7x_2 - 4x_3 &= 2 \end{aligned}$$

$$|||$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \end{array}\right] \xrightarrow{R_2 -= 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \end{array}\right]$$

Eg: Let's solve the system from before using augmented matrices:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ 2x_1 - 3x_2 + 2x_3 = 14 \\ 3x_1 + x_2 - x_3 = -2 \end{cases} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{array}\right] \overset{R_2 -= 2R_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 3 & 1 & -1 & -2 \end{array}\right]$$

$$\overset{R_3 -= 3R_1}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & -5 & -10 & -20 \end{array}\right]$$

$$\overset{R_3 -= \frac{5}{7}R_2}{\rightsquigarrow} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{array}\right]$$

$$\rightsquigarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 6 \\ -7x_2 - 4x_3 = 2 \\ -\frac{50}{7}x_3 = -\frac{150}{7} \end{cases}$$

Now use back-substitution like before.

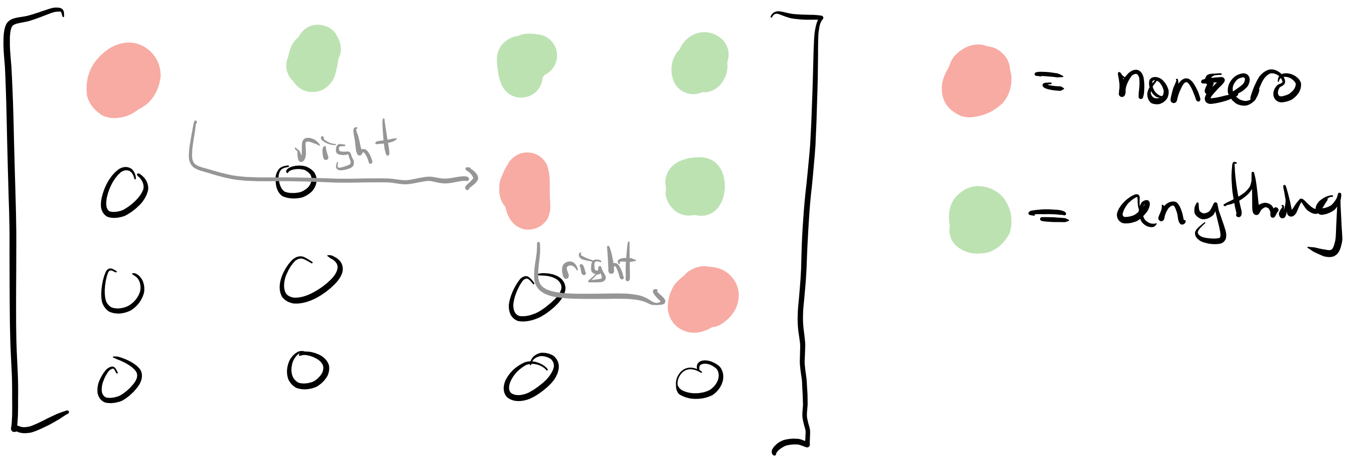
What does it mean to be "done"?

(in terms of augmented matrices)

Def: A matrix is in **row echelon form (REF)** if

(1) The first nonzero entry of each row is to the right of the row above it

(2) All zero rows are at the bottom



REF:
$$\begin{bmatrix} 1 & 1 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & -7 & -4 & 2 \\ 0 & 0 & -\frac{50}{7} & -\frac{150}{7} \end{bmatrix}$$

Not REF:
$$\begin{bmatrix} 1 & 2 & -1 & 4 \\ 2 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 3 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Important: When checking if an augmented matrix is in REF, ignore the augmentation line.

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{array}\right] \text{ REF? } \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 0 & 3 & 12 \end{bmatrix} \checkmark$$

(↖ delete)

Upshot: The elimination procedure **terminates** when your (augmented) matrix is in **REF.**

Solving a linear system $\equiv$ Putting an augmented matrix into REF using row operations

Def: The ***pivot positions (pivots)*** of a matrix are the positions of the 1^{st} nonzero entries of each row *after* you put it into REF.

$$\begin{bmatrix} \boxed{1} & 1 & -1 & 4 \\ 0 & 0 & \boxed{3} & 12 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 2 & 3 & 6 \\ 0 & \boxed{-7} & -4 & 2 \\ 0 & 0 & \boxed{-\frac{50}{7}} & -\frac{150}{7} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-3} & -6 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

● = pivots

Remarkably, this is well-defined!

Def: The rank of a matrix is the number of pivots it has (in REF).

Eg:

$$\begin{bmatrix} 1 & 2 & 3 & 6 \\ 2 & -3 & 2 & 14 \\ 3 & 1 & -1 & -2 \end{bmatrix} \xrightarrow[\text{(p.9)}]{\text{REF}} \begin{bmatrix} \boxed{1} & 2 & 3 & 6 \\ 0 & \boxed{-7} & -4 & 2 \\ 0 & 0 & \boxed{-\frac{50}{7}} & -\frac{150}{7} \end{bmatrix}$$

rank = 3

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & -1 \end{bmatrix} \xrightarrow[\text{(p.4)}]{\text{REF}} \begin{bmatrix} \boxed{1} & 2 & 3 & 1 \\ 0 & \boxed{-3} & -6 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

rank = 2