Inverse Matrices

Question: When solving $Ax=b$, when can we "divide by A"?

If "$x=\frac{b}{A}$" makes sence, then $Ax=b$ has exactly one solution "$x=\frac{b}{A}$" for every b.

This means RREF(A|b) looks like this:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right)$$

Def: An $n\times n$ (square!) matrix A is invertible if there exists another $n\times n$ matrix B such that $AB = I_n = BA$.

$I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ $n\times n$ identity matrix

Otherwise it's called singular.

Notn: $B = A^{-1}$, called the inverse of A.

Eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ $\quad B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow B = A^{-1}$$

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{\text{"B"}} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \neq I_2$

so A is singular (non-invertible).

Remark: Since $AB \neq BA$ in general, you have to require $AB = I_n = BA$ a priori. But:

Fact: If A and B are $n\times n$ matrices and $AB = I_n$ or $BA = I_n$, then $B = A^{-1}$.

So the definition above is a bit pedantic...

Remark: A non-square matrix does not admit both a left- and right-inverse, so not invertible. (Can't solve $AB = I_m$ and $CA = I_n$ unless A is square.)

This is why we only treat invertibility of square matrices.

Fact: $(A^{-1})^{-1} = A$

because $AB = I_n$ means $B = A^{-1}$ and $A = B^{-1}$

Fact: If A & B are invertible, then so is AB, and $(AB)^{-1} = \cancel{A^{-1}B^{-1}}\ B^{-1}A^{-1}$.

Check: $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = A(I_n)A^{-1} = AA^{-1} = I_n$ ✓

NB: Why not $(AB)^{-1} = A^{-1}B^{-1}$? Let's check:

$$(AB)(A^{-1}B^{-1}) = ABA^{-1}B^{-1} = ???$$

(no cancellation – can't re-order the terms!)

Thm: Let A be an $n\times n$ matrix. The Following Are Equivalent: (TFAE) either all are true or all are false

(1) A is invertible

(2) The RREF of A is I_n coefficient matrix!

(3) A has a pivot in every row/every column.
(A has n pivots)

We'll see why a bit later.

Eg: $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$ invertible

● = pivots

Eg: $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is in RREF, $\neq I_2$ singular

How do you compute the inverse?

Algorithm (Matrix Inversion):

Input: A square matrix.

Output: The inverse matrix, or "singular"

Procedure:

(a) Form the augmented matrix $[A \mid I_n]$

(b) Run Gauss-Jordan on $[A \mid I_n]$.

(c) If the output is $[I_n \mid B]$ then $B = A^{-1}$.
Otherwise A is singular.

why does this work? See below.

Eg: Compute $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 -= \frac{1}{2}R_1} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1/2 & -1/2 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \times= 2} \left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 -= 3R_2} \left[\begin{array}{cc|cc} 2 & 0 & 4 & -6 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{R_1 \div= 2} \left[\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

This has the form $\left[\begin{array}{c|cc} I_2 & 2 & -3 \\ & -1 & 2 \end{array}\right]$

So $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$.

Eg: Compute $\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}^{-1}$.

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ -1 & -3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 += R_1} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 -= R_2} \left[\begin{array}{cc|cc} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

But this is in RREF and it does not have the form $[I_2 \mid B] \implies$ singular.

NB: We knew this after the first step: no pivot in the second column.

Actually there's a shortcut for 2×2 matrices:

Fact: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\iff ad-bc \neq 0$, in which case

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Eg: $$\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}^{-1} = \frac{1}{2\cdot 2-3\cdot 1}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

Check: $$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & bd-bd \\ ac-ac & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

What is this good for?

Suppose A is invertible. Let's solve $Ax=b$.

$$Ax=b \iff A^{-1}(Ax)=A^{-1}b$$

$$\iff (A^{-1}A)x = A^{-1}b$$

$$\iff I_n x = A^{-1}b \iff x = A^{-1}b$$

For invertible A:

$$\boxed{Ax=b \iff x=A^{-1}b}$$

In particular, $Ax=b$ has exactly one solution for any b, and we have an expression for b in terms of x

Eg: Solve
$$\begin{aligned} 2x_1+3x_2&=b_1 \\ x_1+2x_2&=b_2 .\end{aligned}$$

$$A=\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad A^{-1}=\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=A^{-1}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}=\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\iff \begin{aligned} x_1 &= 2b_1-3b_2 \\ x_2 &= -b_1+2b_2 \end{aligned}$$

So if you want to solve
$$\begin{aligned} 2x_1+3x_2&=3 \\ x_1+2x_2&=4 \end{aligned}$$

$$\Rightarrow \begin{aligned} x_1&=2(3)-3(4)=-6 \\ x_2&=-(3)+2(4)=5 \end{aligned}$$

Elementary Matrices

These give a way to do row operations by matrix multiplication!

Def: An elementary matrix is a matrix obtained from I_n by doing one row operation.

Eg:

- $R_1 += 2R_2$ $\quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $R_2 \times= 2$ $\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

- $R_1 \leftrightarrow R_2$ $\quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Fact: If E is an $m \times m$ elementary matrix and A is an $m \times n$ matrix, then

$$E \cdot A = \left(\begin{array}{l}\text{what you get by performing that} \\ \text{row operation on } A.\end{array}\right)$$

$$\boxed{(\text{row operations}) \equiv \left(\begin{array}{l}\text{left-multiplication by} \\ \text{elementary matrices}\end{array}\right)}$$

Eg: $\begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 += 2R_2} \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 += 2R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

same!

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & -3 & 0 & -7 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Left-multiplication by $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ does $R_1 += 2R_2$.

Fact: An elementary matrix is invertible. Its inverse corresponds to the elementary matrix that un-does the row operation.

Why? $E_1 = (\text{matrix for } R_1 += 2R_2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = (\text{matrix for } R_1 -= 2R_2) = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_1 E_2 = E_1 E_2 I_n = E_1(\text{do } R_1 -= 2R_2 \text{ to } I_n)$

$= \left(\begin{array}{l} \text{first do } R_1 -= 2R_2 \text{ to } I_n, \\ \text{then do } R_1 += 2R_2 \text{ to } I_n \end{array}\right) = I_n.$

Check: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

What if you do multiple row operations?

Consider these row operations & their elementary matrices:

$E_1: R_1 += 2R_2 \qquad E_2: R_2 \times= 2 \qquad E_3: R_2 \leftrightarrow R_3$

Apply in order to A

$$A \xrightarrow{R_1 += 2R_2} E_1A \xrightarrow[\text{do } R_2\times=2 \text{ to } E_1A]{R_2\times=2} E_2(E_1A) \xrightarrow{R_2 \leftrightarrow R_3} E_3(E_2(E_1A))$$

$$= (E_3E_2E_1)A$$

The elementary matrices ended up in the opposite order! Why?

$E_3E_2E_1A = E_3E_2(E_1A)$: first multiply by E_1 then by E_2 then E_3

Application to Invertibility:

Suppose $RREF(A) = I_n$.

So there are some number of row ops to transform $A \rightsquigarrow I_n$.

Let $E_1, \ldots, E_r$ be their elementary matrices.

$$I_n = (E_r E_{r-1} \cdots E_1) A$$

$$\Rightarrow \boxed{A^{-1} = E_r E_{r-1} \cdots E_1}$$

In particular, A is invertible: justifies (part of) the Thm above (p. 2).

This also justifies the algorithm for computing A^{-1}:

$$[A \mid I_n] \xrightarrow{\text{row ops}} [I_n \mid B]$$

Then $[I_n \mid B] = (E_r \cdots E_1)[A \mid I_n]$

$= [(E_r \cdots E_1)A \mid (E_r \cdots E_1)]$

column ~ 1st matrix multiplication

$\Downarrow$

$C[A|B] = [CA|CB]$

$$\Rightarrow B = E_r \cdots E_1 = A^{-1}$$

Triangular Matrices

Will lead to LU decomposition ⇝ computationally efficient way to solve $Ax=b$ for many values of b.

Def: A matrix is **upper/lower triangular** if all entries below/above the diagonal are zero.

upper triangular

$$\begin{bmatrix} \bullet & \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet & \bullet \\ 0 & 0 & \bullet & \bullet \end{bmatrix}$$

lower triangular

$$\begin{bmatrix} \bullet & 0 & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & \bullet & 0 \end{bmatrix}$$

$\bullet$ = anything

A matrix is **unitriangular** if it is triangular and all diagonal entries = 1.

upper unitriangular

$$\begin{bmatrix} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \end{bmatrix}$$

lower unitriangular

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \bullet & 1 & 0 & 0 \\ \bullet & \bullet & 1 & 0 \end{bmatrix}$$

NB: A matrix is **diagonal** $\iff$ it is both upper- and lower-triangular.

Eg: A matrix in REF is upper-Δular

$$\begin{pmatrix} 1 & 7 & 2 & 4 \\ 0 & 0 & 3 & 9 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eg: If E is the elementary matrix for $R_i += c \cdot R_j$ for $i > j$ (add a higher row to a lower row) then E is lower-uniΔular.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 += 2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E$$

Fact: If A & B are $n \times n$ upper (uni)Δular matrices then so are AB and A^{-1} (if A is invertible).

Likewise for lower (uni) Δular.

Eg: $$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 6 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 7 & 1 & 0 \\ 29 & 11 & 1 \end{pmatrix}$$