Parametric Form

Now we deal systematically with systems of equations with ∞ solutions. We want to parameterize all solutions.

Eg:
$$2x + y + 12z = 1$$
$$x + 2y + 9z = -1$$

$$\longrightarrow \left[\begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right]$$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right] \rightsquigarrow \begin{array}{l} x + 5z = 1 \\ y + 2z = -1 \end{array}$$

Observation: If you substitute **any** number for z, you get the system

$$\text{unknowns} \rightarrow \begin{array}{l} x = 1 - 5z \leftarrow \\ y = -1 - 2z \leftarrow \end{array} \text{numbers}$$

which has a unique solution!

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix} \qquad \text{eg} \quad z = 1: \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \\ 1 \end{pmatrix}$$

check: $2(-4) + (-3) + 12(1) = 1$
$-4 + 2(-3) + 9(1) = -1$ ✓

this is the **parametric form** of the solution; z is the **free variable** or **parameter**.

Implicit vs Parameterized Form.

The system of equations
$$\begin{cases} 2x + y + 12z = 1 \\ x + 2y + 9z = -1 \end{cases}$$
are **implicit equations** of a line: it expresses the line as the set of **solutions** of these equations without giving you any way to write down specific points on the line. The parametric form
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 - 5z \\ -1 - 2z \\ z \end{pmatrix}$$
is a **parametric equation** for the same line: it gives you a way to **produce** all solutions in terms of the **parameter** z.

[demo]
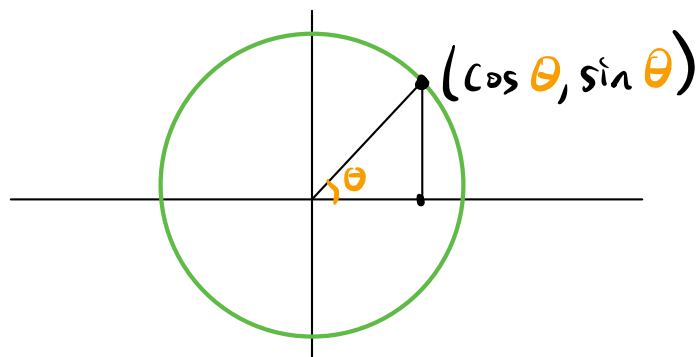
Non-linear example:

An **implicit equation** for the unit circle is
$$x^2 + y^2 = 1$$

A **parametric equation** for the unit circle is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

θ = parameter

$(\cos\theta, \sin\theta)$

Here's how to produce parametric equations for general linear systems.

Recall: A pivot column of a matrix is a column with a pivot.

Def: A free variable in a system of equations is a variable whose column (in the coeff matrix) is not a pivot column.

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right]$$
$$x \quad y \quad z$$

● x, y in pivot cols
● z is free

These are the variables you can't solve for in back-substitution.

Procedure (Parametric Form):
To find the parametric form of the solutions of $Ax = b$:
(1) Put $[A|b]$ into RREF. Stop if inconsistent.
(2) Write out the corresponding equations
(3) Move free variables to the right-hand side
All solutions are obtained by substituting any values for the free variables.

This uses the free variables as the parameters.

So **Gauss-Jordan elimination** turns **implicit equations**
↳ (step (1) above)
into **parametric equations.**

Gauss-Jordan

implicit
equations

parametric
equations

orthogonal
complements (later)
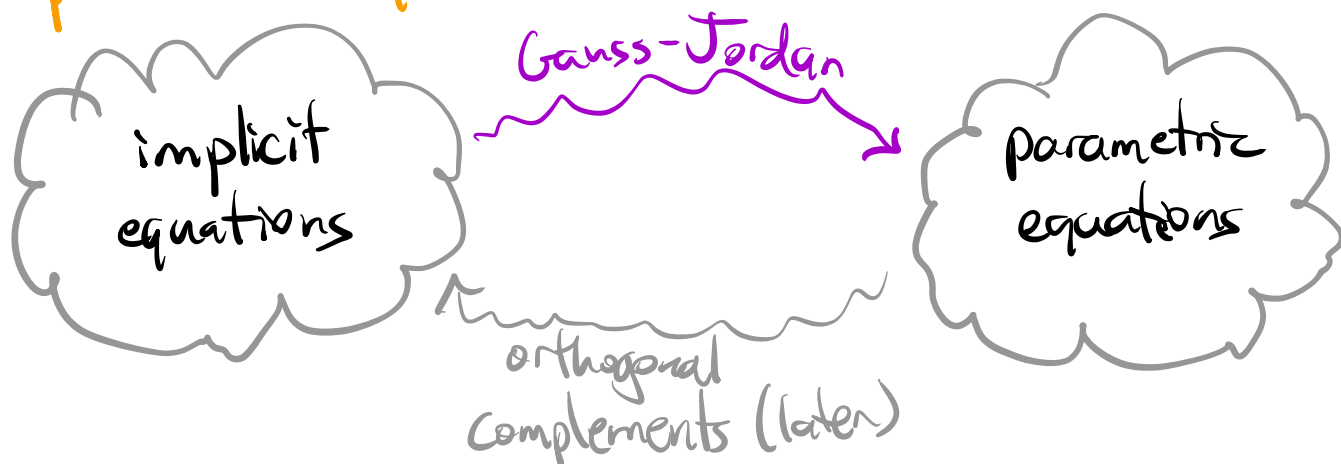
Eg: $x + y + z = 1$ ⟿ $\begin{bmatrix} 1 & 1 & 1 & | & 1 \end{bmatrix}$

This is already in RREF!
Free variables: y, z

$\xrightarrow[\text{form}]{\text{parametric}}$ $x = 1 - y - z$

This is a parameterized **plane.** [demo]

Eg: $\begin{matrix} x + y = 2 \\ x - y = 0 \end{matrix}$ ⟿ $\begin{bmatrix} 1 & 1 & | & 2 \\ 1 & -1 & | & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 1 \end{bmatrix}$

No free variables! Just have one solution
$x = 1, y = 1.$

Observation:

- 2 free variables / 2 parameters:
  solution set is a plane

- 1 free variable / 1 parameter:
  solution set is a line

- 0 free variables / 0 parameters:
  solution set is a point

Provisional Def^n: The dimension of the solution set of a consistent system $Ax = b$ is the number of free variables.

Parametric Vector Form

This is an alternate, more concise way of writing a solution set in parametric form.

Eg: $2x + y + 12z = 1$
$x + 2y + 9z = -1$
$\xrightarrow[\text{form}]{\text{parametric}}$
$x = 1 - 5z$
$y = -1 - 2z$
$z = z$

(from before)

parameterize the free variable too

Let's rewrite this as **one** equation involving **vectors**:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

This is the **line** thru $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ in the $\begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$ - direction.   [demo again]

note nice columns

Eg: $x + y + z = 1$
$\xrightarrow[\text{form}]{\text{parametric}}$
$x = 1 - y - z$
$y = y$
$z = z$

} parameterize the free variables

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \boxed{y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

linear combination of $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

This is the **plane** containing $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, & $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$
      [demo again]

Writing the solution set in this way is called the parametric vector form.

Procedure (Parametric Vector Form)
   To find the parametric vector form of the solutions of $Ax = b$:

   (1 – 3) Find the parametric form

   (4) Add trivial equations for the free variables, in order. Organize the right-hand side into columns.

   (5) Gather the columns into vectors. Pull out the free variables as coefficients.

Result:
$$X = \begin{pmatrix} a \ \text{constant} \\ \text{vector} \end{pmatrix} + \begin{pmatrix} a \ \text{linear combination with} \\ \text{the free variables as weights} \end{pmatrix}$$

NB: The constant vector is the solution you get by setting all free variables $= 0$.

Def: This vector is called a particular solution.
   (It is a solution of $Ax = b$)

Eg:
$$x + 2y + 2z + w = 1$$
$$2x + 4y + z - w = -1$$

$$\rightsquigarrow \begin{bmatrix} 1 & 2 & 2 & 1 & | & 1 \\ 2 & 4 & 1 & -1 & | & -1 \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 & | & -1 \\ 0 & 0 & 1 & 1 & | & 1 \end{bmatrix}$$

$$\rightsquigarrow \quad x + 2y \quad - w = -1$$
$$z + w = 1$$

free

$$\rightsquigarrow \begin{cases} x = -1 - 2y + w \\ y = \qquad\qquad y \\ z = 1 \qquad\qquad - w \\ w = \qquad\qquad w \end{cases} \quad \leftarrow \text{trivial equations}$$

columns

$$\xrightarrow{\text{PVF}} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

particular solution

any linear combination of $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

Vector Equations

This is another way of writing a linear system that works well with what we've been doing.

Def: A vector equation is an equation involving linear combinations of vectors with unknown coefficients.

Eg:
$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

This is equivalent to the system
$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(use the column-first definition of the matrix-vector product). But now we're thinking in terms of linear combinations of vectors.

→ we will draw **pictures** of these (next time)

Four Ways to Write a System of Eqns:

(1) Linear system

$$x_1 - x_2 = 8$$
$$2x_1 - 2x_2 = 16$$
$$6x_1 - x_2 = 3$$

(2) Matrix Equation

$$\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

(columns)

(3) Augmented Matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right)$$

(4) Vector equation

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

You still **solve** a vector equation by putting it into an augmented matrix:

Eg: $\quad x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \quad \rightsquigarrow \quad \left[\begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right]$

$$\xrightarrow{\text{RREF}} \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array} \right]$$

Solution is $\quad x_1 = -1, \quad x_2 = -9$

Important Observation: (!!!!)

$$x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \qquad \text{has a solution (consistent)}$$

$$\iff \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \text{ is a } \textbf{linear combination} \text{ of } \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$$

in which case the solution $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is the vector of **weights**

In fact, we know $x_1 = -1$, $x_2 = -9$:

$$-1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} - 9 \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} = \begin{pmatrix} \text{linear combination} \\ \text{of } \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \& \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \end{pmatrix}$$

different b

Eg: $x_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + x_2 \begin{pmatrix} -1 \\ -3 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 & | & 2 \\ 2 & -3 & | & -2 \\ 6 & -1 & | & 0 \end{bmatrix}$

$\overset{REF}{\rightsquigarrow} \begin{bmatrix} 1 & -1 & | & 2 \\ 0 & 5 & | & -12 \\ 0 & 0 & | & -6 \end{bmatrix}$ inconsistent

So, $\begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$ is **not** a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} \& \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$.

In general:

next time

Column Picture Criterion for Consistency

$Ax = b$ is **consistent** (has at least one solution)

$\Updownarrow$

b is a **linear combination** of the **columns** of A
in which case $x =$ the weights.

We now have 2 ways that linear combinations appear when solving a system of equations:

Linear Systems & Linear Combinations

(1) $Ax = b$ is consistent $\iff$ b is a linear combination of the columns of A.

(2) In this case, the solution set has the form
$$x = \begin{pmatrix} \text{particular} \\ \text{solution} \end{pmatrix} + \begin{pmatrix} \text{all} & \text{linear combinations} \\ \text{of a set of vectors} \end{pmatrix}$$

Next time: Spans: this is what the set of all linear combinations of a list of vectors looks like.