The Four Subspaces

Recall: To any matrix A, we can associate:

- $\mathrm{Col}(A)$; basis = pivot columns of A
- $\mathrm{Nul}(A)$; basis = vectors in the PVF of $Ax=0$

There are two more subspaces: just replace A by A^T, then take Col & Nul.

Why? Orthogonality (next time...)

Def: The row space of A is $\mathrm{Row}(A) = \mathrm{Col}(A^T)$.

This is the subspace spanned by the rows of A, regarded as vectors in $\mathbb{R}^n$.

This is a subspace of $\mathbb{R}^n$ $\quad n = \#\text{columns}$

($n = \#$entries in each row)

$\rightsquigarrow$ row picture

Eg: $$\mathrm{Row}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \mathrm{Span}\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\right\}$$
$$= \mathrm{Col}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$$

Fact: Row operations do not change the row space.

Why? If the rows are v_1, v_2, v_3 then $\mathrm{Row}(A) = \mathrm{Span}\{v_1, v_2, v_3\}$. Row ops:

- $R_1 \longleftrightarrow R_3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_3, v_2, v_1\}$
- $R_2 \times\!= 3$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, 3v_2, v_3\}$
- $R_2 \mathrel{+}= 2R_1$: $\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1, v_2+2v_1, v_3\}$

  because $v_2 + 2v_1 \in \mathrm{Span}\{v_1, v_2, v_3\}$

  and $v_2 = (v_2+2v_1) - 2v_1 \in \mathrm{Span}\{v_1, v_2+2v_1, v_3\}$ ✓

This is a col space (of A^T), so you know how to compute a basis (pivot columns of A^T). But you can also find a basis by doing elimination on A:

Thm: The nonzero rows of a REF of A form a basis for $\mathrm{Row}(A)$.

Eg:
$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \xrightarrow{\text{REF}} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis: $\left\{ \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ -3 \end{pmatrix} \right\}$

Proof: by "forward-substitution":

(1) Spans: row ops don't change Row(A), and you can always delete the zero vector without changing the span

(2) LI: $0 = x_1\begin{pmatrix}1\\2\\2\\1\end{pmatrix} + x_2\begin{pmatrix}0\\0\\-3\\-3\end{pmatrix} = \begin{pmatrix}x_1\\2x_1\\2x_1 - 3x_2\\x_1 - 3x_2\end{pmatrix}$

● = pivot, so this entry in the sum is just $(1)\cdot x_1 = 0 \Rightarrow x_1 = 0$

● = pivot, so this entry in the sum is just $(-3)x_2 = 0 \Rightarrow x_2 = 0$ ✓

Consequence: $\dim \mathrm{Row}(A) = \#$ pivot rows $= \#$ pivots $=$ rank.

(a nonzero row of an REF matrix has a pivot)

Def: The left null space of A is $\mathrm{Nul}(A^T)$.

This is the solution set of $A^Tx = 0$.

Notation: just $\mathrm{Nul}(A^T)$ (no new notation)

This is a subspace of $\mathbb{R}^m$ $m = \#$rows

($m = \#$columns of A^T)

$\rightsquigarrow$ column picture

NB: $A^Tx = 0 \iff 0 = (A^Tx)^T = x^TA$

so $Nul(A^T) = \{$row vectors $y \in \mathbb{R}^m$: $yA = 0\}$

$Nul(A^T)$ is a null space, so you know how to compute a basis (PVF of $A^Tx=0$). You can also find a basis by doing elimination on A:

Thm: If $EA=U$ for E = product of elementary $m\times m$ matrices and U a matrix in REF, and if U has $m-r$ zero rows, then the last $m-r$ rows of E form a basis for $Nul(A^T)$.

r = rank = #pivots = #nonzero rows

Consequence: $\dim Nul(A^T) = m-r$ = #rows − rank

Where did E come from? Elementary matrices! Doing row ops means left-multiplying by these:

$$A \rightsquigarrow E_1A \rightsquigarrow E_2E_1A \rightsquigarrow E_3E_2E_1A = U$$

So $EA=U$ for $E=E_3E_2E_1$, which is the matrix you get by doing the same row ops on I_m.

Procedure: To compute a basis of $\mathrm{Nul}(A^T)$:

(1) Form the augmented matrix $(A \mid I_m)$

(2) Eliminate to REF

(3) The rows on the right side of the line next to zero rows on the left form a basis of $\mathrm{Nul}(A^T)$.

Eg:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix}$$

$$\left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & -1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 -= R_1]{R_2 -= 2R_1} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 -= R_2} \left[\begin{array}{cccc|ccc} 1 & 2 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & -3 & -3 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right]$$

zero row basis

Basis for $\mathrm{Nul}(A^T)$: $\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Check: $$\begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \checkmark$$

Proof of the Thm: If U is in REF and the last $m-r$ cols are zero then

$$\mathrm{Nul}(U^T) = \mathrm{Span}\{e_{m-r+1}, e_{m-r+2}, \ldots, e_m\}:$$

This is because $U^T e_i =$ the i^{th} row of U. We know from before that the nonzero rows of U are LI. And $U^T e_{m-r+i} =$ a zero row, so $e_{m-r+i} \in \mathrm{Nul}(U^T)$.

But $U = EA$, so $U^T = A^T E^T$, and

$$0 = U^T e_{m-r+i} = A^T E^T e_{m-r+i}$$

$$= A^T (m-r+i^{th} \text{ row of } E).$$

NB: The left null space is changed by row operations:

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \\ 1 & 2 & -1 & -2 \end{bmatrix} \qquad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}$$

$\rightsquigarrow$

$$U = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathrm{Nul}(U^T) = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

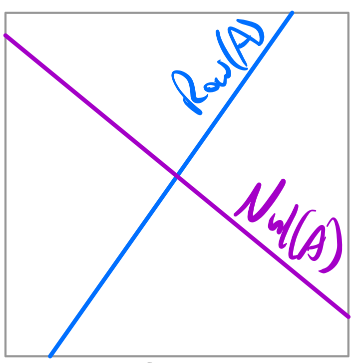
Summary: Four Subspaces

A : an $m \times n$ matrix of rank r

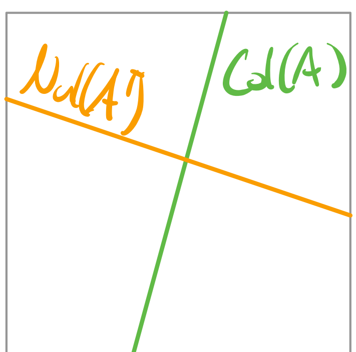
Subspace	of	row/col	dim	basis
$\mathrm{Col}(A)$	$\mathbb{R}^m$	col	r # pivot cols	pivot cols of A
$\mathrm{Nul}(A)$	$\mathbb{R}^n$	row	$n-r$ # free vars	vectors in PVF
$\mathrm{Row}(A)$	$\mathbb{R}^n$	row	r # pivot rows	nonzero rows of REF
$\mathrm{Nul}(A^T)$	$\mathbb{R}^m$	col	$m-r$ # zero rows in REF	last $m-r$ rows of E

The row picture subspaces ($\mathrm{Nul}(A)$, $\mathrm{Row}(A)$) are unchanged by row operations.

The col picture subspaces ($\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$) are changed by row operations.



row picture $\mathbb{R}^n$



col picture $\mathbb{R}^m$

Consequences:

$$\boxed{\begin{array}{c}\text{Row Rank} = \text{Column Rank} \\ \dim \mathrm{Row}(A) = \mathrm{rank} = \dim \mathrm{Col}(A)\end{array}}$$

So A & A^T have the same # pivots — in completely different positions! (HW#6)

$$\boxed{\begin{array}{c}\text{Rank-Nullity} \\ \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A) = n = \#\,\mathrm{cols} \\ \dim \mathrm{Row}(A) + \dim \mathrm{Nul}(A^T) = m = \#\,\mathrm{rows}\end{array}}$$

[demos]

NB: You can compute bases for all four subspaces by doing elimination once.

$$A \rightsquigarrow [A \mid I_m] \rightsquigarrow [\mathrm{RREF}(A) \mid E]$$

- Get the pivots of $A \rightsquigarrow \mathrm{Col}(A)$
- Get $\mathrm{RREF}(A) \rightsquigarrow$ PVF of $Ax=0 \rightsquigarrow \mathrm{Nul}(A)$
- Get nonzero rows of $\mathrm{RREF}(A) \rightsquigarrow \mathrm{Row}(A)$
- Get rows of $E \rightsquigarrow \mathrm{Nul}(A^T)$

Full-Rank Matrices

A "random" matrix will have largest rank possible.
This is an important special case.

Def: An $m \times n$ matrix A of rank r has:

- full column rank if $r = n$ eg. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
- full row rank if $r = m$ eg. $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

NB: Each row & column has at most one pivot

so $r \leq \min\{m, n\}$

Hence full row/column rank means full rank

ie. largest possible rank.

NB: A has full column rank $\Rightarrow n = r \leq m$

$\Rightarrow A$ is tall (at least as many rows as cols)

A has full row rank $\Rightarrow m = r \leq n$

$\Rightarrow A$ is wide (at least as many cols as rows)

We've seen several properties of matrices that translate into "there's a pivot in every column".

Thm: The Following Are Equivalent (TFAE):

(for a given matrix A, all are true or all are false)

(1) A has full column rank

  (1') A has a pivot in every column

  (1'') A has no free columns.

(2) $\mathrm{Nul}(A) = \{0\}$

  (2') $Ax = 0$ has only the trivial solution.

☆ (2'') $Ax = b$ has 0 or 1 soln for every $b \in \mathbb{R}^m$

(3) The columns of A are LI

(4) $\dim \mathrm{Col}(A) = n$

(5) $\dim \mathrm{Row}(A) = n$

  (5') $\mathrm{Row}(A) = \mathbb{R}^n$

We've seen several properties of matrices that translate into "there's a pivot in every row":

Thm: TFAE:

(1) A has full row rank

(1') A has a pivot in every row

(1'') A REF of A has no zero rows

(2) $\dim \mathrm{Col}(A) = m$

(2') $\mathrm{Col}(A) = \mathbb{R}^m$

☆ (2'') $Ax = b$ is consistent for every $b \in \mathbb{R}^m$

(3) The columns of A span $\mathbb{R}^m$

(4) $\dim \mathrm{Row}(A) = m$

(5) $\mathrm{Nul}(A^T) = \{0\}$