Invertibility Revisited

Recall:

A has **full column rank** if it has a pivot in every column ($r=n$),

A has **full row rank** if it has a pivot in every row ($r=m$)

If A has full column rank **and** full row rank then

$$n=r=m$$

$\Rightarrow$ A is **square** and has n pivots: **invertible.**

Thm: For an **$n \times n$** matrix A, **TFAE:**

(1) A is invertible

(2) A has full column rank

(3) A has full row rank

(4) $RREF(A)=I_n$

(5) There is a matrix B with $AB=I_n$

(6) There is a matrix B with $BA=I_n$

☆ (7) $Ax=b$ has **exactly one solution** for **every b**

↳ namely, $x=A^{-1}b$

(8) A^T is invertible

↳ (row rank = col rank)

Eg: If A is invertible then its columns

- span $\mathbb{R}^n$ (full row rank)
- are LI (full col rank)

$\Rightarrow$ basis for $\mathbb{R}^n$

Conversely, any basis for $\mathbb{R}^n$ are the columns of an invertible matrix

- spans $\Rightarrow$ full row rank
- LI $\Rightarrow$ full col rank

$$\boxed{\text{Basis of } \mathbb{R}^n \equiv \text{cols of an invertible } n\times n \text{ matrix}}$$

So $\mathbb{R}^n$ has many bases! (not just $\{e_1, \ldots, e_n\}$)

NB: for an $n\times n$ matrix,

full col rank $\Longleftrightarrow$ invertible $\Longleftrightarrow$ full row rank

In terms of columns, n vectors in $\mathbb{R}^n$

spans $\mathbb{R}^n$ $\Longleftrightarrow$ linearly independent

This is a special case of the basis theorem.

Basis Theorem: Let V be a subspace of dim d

(1) If d vectors span V then they're a basis

(2) If d vectors in V are LI then they're a basis.

So if you have the correct number of vectors, you only need to check one of spans / LI.

Eg: • Two noncollinear vectors in a plane form a basis.

• Two vectors that span a plane form a basis.

Geometry of Dot Products

Recall: $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $w = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \rightarrow v \cdot w = x_1 y_1 + \cdots + x_n y_n = v^T w$

Dot products measure length & angles

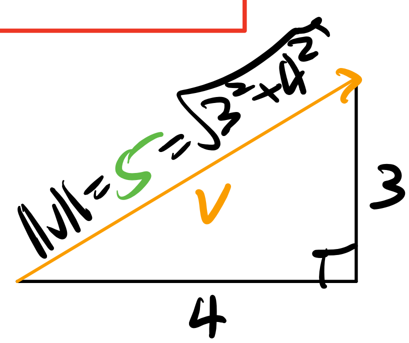
→ geometric questions about length & angles become algebraic questions about dot products.

Recall: If $v = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ then
$$v \cdot v = x_1^2 + x_2^2 + \cdots + x_n^2 \geq 0$$

Def: The length of v is

$$\boxed{\|v\| = \sqrt{v \cdot v} \quad \text{ie} \quad \|v\|^2 = v \cdot v}$$

This makes sense by the Pythagorean theorem: $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



Sanity Check: $c \in \mathbb{R}$ $\quad v \in \mathbb{R}^n$

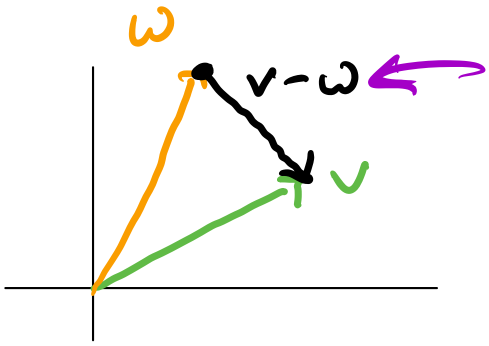
$$\|cv\| = \left\| c\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right\| = \left\| \begin{pmatrix} cx_1 \\ \vdots \\ cx_n \end{pmatrix} \right\| = \sqrt{(cx_1)^2 + \cdots + (cx_n)^2}$$

$$= |c| \cdot \sqrt{x_1^2 + \cdots + x_n^2} = |c| \cdot \|v\| \quad \checkmark$$

Eg: $2v$ is twice as long as v.

So is $-2v$.

Def: The distance from v to w is $\|v-w\| = \|w-v\|$



length of $v-w$ is distance from v to w

Def: A unit vector is a vector of length 1.

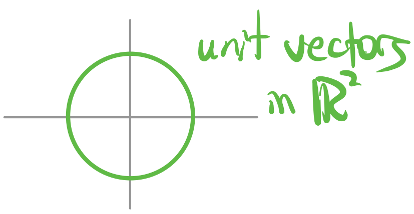
ie $\|v\| = 1$ $\quad$ ie. $\|v\|^2 = v \cdot v = 1$

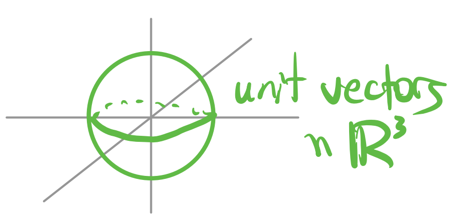
If $v = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ then v is a unit vector

$\iff x_1^2 + \cdots + x_n^2 = 1$

$\iff v$ lies on the unit $(n-1)$-sphere

($n=2$: unit circle)





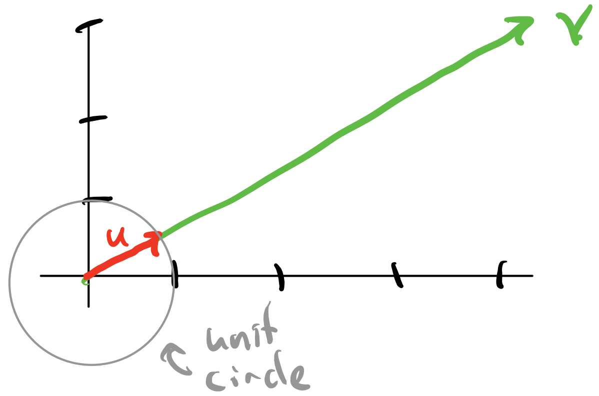
If $v \neq 0$, the unit vector in the direction of v is the vector

$$u = \frac{1}{\|v\|} \cdot v = \frac{v}{\|v\|} \qquad (\text{scalar} \times \text{vector})$$

NB: $\|u\| = \left| \frac{1}{\|v\|} \right| \cdot \|v\| = \frac{\|v\|}{\|v\|} = 1$ ✓

Eg: $v = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ $\quad \|v\| = \sqrt{3^2 + 4^2} = 5$

$$u = \frac{1}{\|v\|} v = \frac{1}{5} \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4/5 \\ 3/5 \end{pmatrix}$$

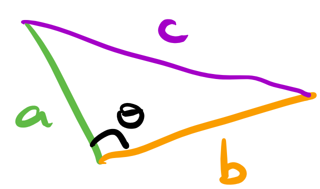


NB: all unit vectors in $\mathbb{R}^2$ are on the unit circle.

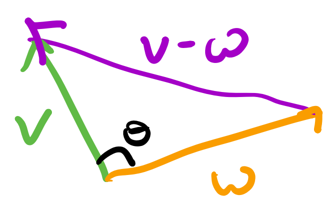
What about $v \cdot w$ for $v \neq w$?

Law of Cosines:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$



Vector Version:



$$\|v-w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$$

$(a = \|v\| \quad b = \|w\| \quad c = \|v-w\|)$

Algebra:

"left hand side" LHS: $\|v-w\|^2 := (v-w)\cdot(v-w)$

FOIL

$= v\cdot v + w\cdot w - 2v\cdot w$

$= \|v\|^2 + \|w\|^2 - 2v\cdot w$

"right hand side" RHS $= \|v\|^2 + \|w\|^2 - 2\|v\|\|w\|\cos\theta$

cancel ⤳

$$\boxed{v\cdot w = \|v\|\|w\|\cos\theta \quad \text{or} \quad \cos\theta = \frac{v\cdot w}{\|v\|\|w\|} \quad (\text{if } v,w\neq 0)}$$

Def: The angle from v to w $(v,w\neq 0)$ is

$$\theta := \cos^{-1}\left(\frac{v\cdot w}{\|v\|\|w\|}\right)$$

NB: $|\cos\theta| = \left|\frac{v\cdot w}{\|v\|\|w\|}\right| \in [0,1]$

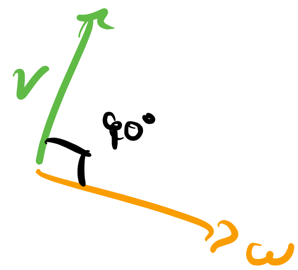
$$\Rightarrow \quad |v\cdot w| \leq \|v\|\cdot\|w\|$$

Schwartz Inequality: $|v\cdot w| \leq \|v\|\cdot\|w\|$ ✓

Def: Vectors v and w are orthogonal or perpendicular, written $v\perp w$, if $\boxed{v\cdot w = 0}$

This says that either:

- $v=0$ or $w=0$ (or both), or
- $\cos(\theta)=0 \iff \theta=\pm 90°$



NB: The zero vector is orthogonal to every vector:
$$0\cdot v=0 \quad \text{for all } v$$

Orthogonality

We are now aiming to find the "best" approximate solution of $Ax=b$ when no actual solution exists.

Eg: find the best-fit ellipse through these points from the 1st lecture...

Q: How close can Ax get to b?
$$\mathrm{Col}(A)=\{Ax : x\in\mathbb{R}^n\}$$
so this means: what is the closest vector $\hat{b}$ in $\mathrm{Col}(A)$ to b?

A: $b-\hat{b}$ is perpendicular to $\mathrm{Col}(A)$

[demo]

So we want to understand what vectors are perpendicular to a subspace.

Eg: Find all vectors orthogonal to $v = \begin{pmatrix}1\\1\\1\end{pmatrix}$.

We need to solve $v \cdot x = 0$

$$\Leftrightarrow v^T x = 0$$

This is just $\mathrm{Nul}(v^T)$:

$$\begin{bmatrix}1 & 1 & 1\end{bmatrix} \rightsquigarrow x_1 + x_2 + x_3 = 0$$

$$\overset{PF}{\rightsquigarrow} \quad \begin{aligned} x_1 &= -x_2 - x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

$$\overset{PVF}{\rightsquigarrow} \quad x = x_2\begin{pmatrix}-1\\1\\0\end{pmatrix} + x_3\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

[demo] $\rightsquigarrow \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$ a plane

Check: $\begin{pmatrix}-1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \quad \begin{pmatrix}-1\\0\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = 0$ ✓

Eg: Find all vectors orthogonal to $v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ & $v_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

We need to solve $\begin{cases} v_1^T \cdot x = 0 \\ v_2^T \cdot x = 0 \end{cases} \rightsquigarrow \begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 &= 0 \end{aligned}$

Equivalently, $\begin{pmatrix} — v_1^T — \\ — v_2^T — \end{pmatrix} \cdot x = \begin{pmatrix} v_1^T x \\ v_2^T x \end{pmatrix} = 0$

So we want $\mathrm{Nul}\begin{pmatrix} - v_1^T - \\ - v_2^T - \end{pmatrix} = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow{\text{PF}} \begin{aligned} x_1 &= -x_2 \\ x_2 &= x_2 \\ x_3 &= 0 \end{aligned}$$

$$\xrightarrow{\text{PVF}} x = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$\rightsquigarrow \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$$ a line

Check: $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$ ✓

[demo]

NB: If $x \perp v_1$ and $x \perp v_2$ then

$$x \cdot (av_1 + bv_2) = a\, x \cdot v_1 + b\, x \cdot v_2 = a \cdot 0 + b \cdot 0 = 0$$

So x is orthogonal to every vector in $\mathrm{Span}\{v_1, v_2\}$

[demo again]

More generally,

$$\boxed{\left\{ v \in \mathbb{R}^n : \begin{array}{l} v \text{ is orthogonal} \\ \text{to every vector} \\ \text{in Span}\{v_1, \dots, v_n\} \end{array} \right\} = \operatorname{Nul} \begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix}}$$