Orthogonal Complements

Recall: We're aiming to find the best approximate solution of $Ax=b$

$\rightsquigarrow$ solve $Ax=\hat{b}$ where $\hat{b}$ is the closest vector to b in $\mathrm{Col}(A)$

$\rightsquigarrow$ solve $Ax=\hat{b}$ where $\hat{b}-b \perp \mathrm{Col}(A)$ [demo]

Def: Two subspaces V, W of $\mathbb{R}^n$ are **orthogonal** if every vector in V is orthogonal to every vector in W:

$$v \in V \quad w \in W \implies v \cdot w = 0$$

The **orthogonal complement** of V is

$$V^\perp = \left\{ w \in \mathbb{R}^n : \begin{array}{l} w \text{ is orthogonal to} \\ \text{every vector in } V \end{array} \right\}$$

NB: Note the difference in notations:

- $V^\perp$ is the orthogonal complement of a subspc
- A^T is the transpose of a matrix.

NB: If V & W are orthogonal and x is in both V and W then $x \cdot x = 0 \implies x = 0$, so $V \cap W = \{0\}$

intersect

Last time: we showed

$$\boxed{\operatorname{Span}\{v_1, \dots, v_n\}^\perp = \operatorname{Nul}\begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix}}$$

Eg: $V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\} \implies V^\perp = \operatorname{Nul}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$

Eg: $V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \implies V^\perp = \operatorname{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix}$

[demos again]

Eg: $V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ and $W = \operatorname{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$ ← line

are orthogonal subspaces, but not orthogonal complements: $V^\perp$ is the plane

$$V^\perp = \operatorname{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\\1\end{pmatrix}\right\}$$

In general, V & W are orthogonal means W is contained in $V^\perp$ (or V is contained in $W^\perp$) (they need not be equal).

Eg: $\{0\}^\perp = \mathbb{R}^n$ and $(\mathbb{R}^n)^\perp = \{0\}$

Fact: $V^\perp$ is also a subspace of $\mathbb{R}^n$.

Check:

(1) Let $x, y \in V^\perp$. So $x \cdot v = 0$ and $y \cdot v = 0$ for every $v \in V$. So $(x+y) \cdot v = x \cdot v + y \cdot v = 0 + 0$ for every $v \in V$ $\rightsquigarrow$ $x + y \in V^\perp$.

(2) Let $x \in V^\perp$, $c \in \mathbb{R}$. So $x \cdot v = 0$ for every $v \in V$. So $(cx) \cdot v = c(x \cdot v) = c \cdot 0 = 0$ for every $v \in V$ $\rightsquigarrow$ $cx \in V^\perp$.

(3) $0 \cdot v = 0$ for every $v \in V$ $\rightsquigarrow$ $0 \in V^\perp$.

Or:

Every subspace is a span, and the orthogonal complement of a span is a null space (which is a subspace).

Facts: Let V be a subspace of $\mathbb{R}^n$.

(1) $\dim(V) + \dim(V^\perp) = n$

(2) $(V^\perp)^\perp = V$

[demos]

NB: (2) says V and $V^\perp$ are orthogonal complements of each other.

Orthogonality of the Four Subspaces

Recall: If someone gives you a subspace, Step 0 is to write it as a column space or a null space. So we want to understand $\mathrm{Col}(A)^\perp$ & $\mathrm{Nul}(A)^\perp$.

Let $A = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$. Then

$$\mathrm{Col}(A)^\perp = \mathrm{Span}\{v_1, \ldots, v_n\}^\perp = \mathrm{Nul}\begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix} = \mathrm{Nul}(A^T)$$

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T)$$

$$\xrightarrow{\text{Take } (\cdot)^\perp} \quad \mathrm{Col}(A) = (\mathrm{Col}(A)^\perp)^\perp = \mathrm{Nul}(A^T)^\perp$$

$$\xrightarrow[\text{by } A^T]{\text{replace } A} \quad \mathrm{Row}(A) = \mathrm{Col}(A^T) = \mathrm{Nul}(A)^\perp$$

and $\mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$

Orthogonality of the Four Subspaces:

$$\mathrm{Col}(A)^\perp = \mathrm{Nul}(A^T) \qquad \mathrm{Nul}(A^T)^\perp = \mathrm{Col}(A)$$

$$\mathrm{Nul}(A)^\perp = \mathrm{Row}(A) \qquad \mathrm{Row}(A)^\perp = \mathrm{Nul}(A)$$

This says the two row picture subspaces $\mathrm{Row}(A)$, $\mathrm{Nul}(A)$ are orthogonal complements, & the two column picture subspaces $\mathrm{Col}(A)$, $\mathrm{Nul}(A^T)$ are orthogonal complements.

Eg: $V = \left\{ x \in \mathbb{R}^3 : \begin{matrix} x+2y=z \\ x+y+z=0 \end{matrix} \right\}$ What is $V^\perp$?

Step 0: $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix} \rightsquigarrow V^\perp = \mathrm{Row}\begin{pmatrix} 1 & 2 & -1 \\ 1 & 1 & 1 \end{pmatrix}$

$$V^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$: no work needed!

Eg: $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{array}\right)$$

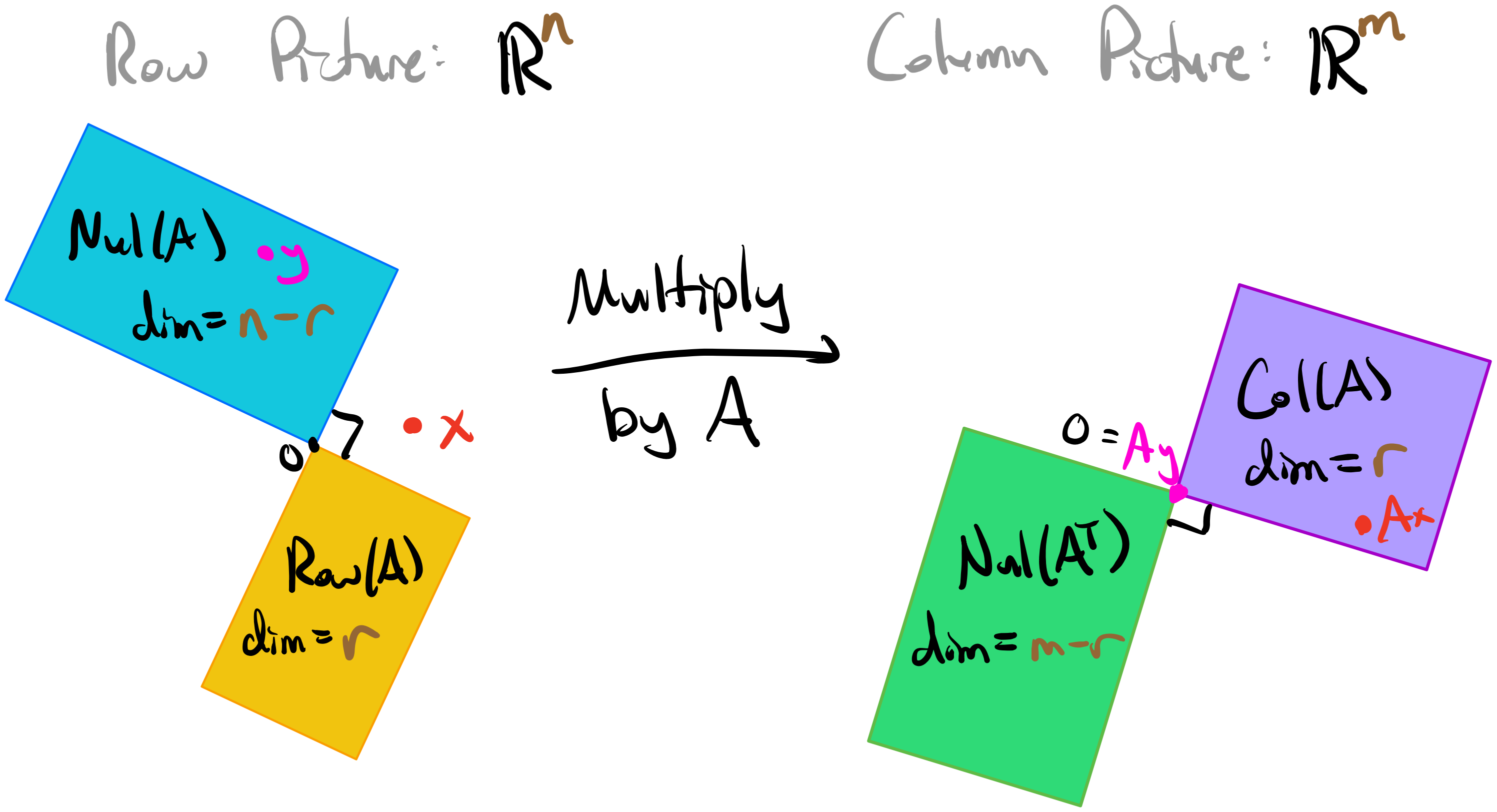
$\rightsquigarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Nul}(A^T) = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

$\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ $\quad \mathrm{Row}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

Here's the picture to have in your head:

The Big Picture

for an $m \times n$ matrix A of rank r



NB: The dimensions match up with $\dim V + \dim V^{\perp} = n$:

$$\dim \mathrm{Nul}(A) + \dim \mathrm{Row}(A) = n$$

$$\dim \mathrm{Nul}(A^T) + \dim \mathrm{Col}(A) = m \quad \checkmark$$

As an application, we can prove an Important Fact that we will use many times:

$$\boxed{\mathrm{Nul}(A^TA) = \mathrm{Nul}(A)}$$

Proof: $\mathrm{Nul}(A^TA)$ contains $\mathrm{Nul}(A)$: HW 5 #8(a)

$x \in \mathrm{Nul}(A) \Rightarrow Ax = 0 \Rightarrow A^TAx = 0 \Rightarrow x \in \mathrm{Nul}(A^TA)$

$\mathrm{Nul}(A)$ contains $\mathrm{Nul}(A^TA)$:

$x \in \mathrm{Nul}(A^TA) \Rightarrow A^TAx = 0 \Rightarrow Ax \in \mathrm{Nul}(A^T)$

$\Rightarrow Ax \in \mathrm{Col}(A)$ and $\mathrm{Nul}(A^T)$

$\Rightarrow (Ax)\cdot(Ax) = 0 \Rightarrow Ax = 0 \Rightarrow x \in \mathrm{Nul}(A)$ ✓

NB: If A has columns $v_1, \ldots, v_n$ then

$$A^TA = \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_n^T — \end{pmatrix}\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} = \begin{pmatrix} v_1\cdot v_1 & v_1\cdot v_2 & \cdots & v_1\cdot v_n \\ v_2\cdot v_1 & v_2\cdot v_2 & \cdots & v_2\cdot v_n \\ \vdots & & \ddots & \vdots \\ v_n\cdot v_1 & v_n\cdot v_2 & \cdots & v_n\cdot v_n \end{pmatrix}$$

This is the matrix of column dot products: the (i,j)-entry is $(\text{col } i)\cdot(\text{col } j)$

Implicit Equations, Revisited

Recall: $\mathrm{Nul}(A) \xrightarrow{\text{PVF}} \mathrm{Span}\{v_1, \ldots, v_{n-r}\}$

takes the implicit equation $Ax = 0$ and generates the parametric form

$$x = a_1 v_1 + \cdots + a_{n-r} v_{n-r}.$$

Orthogonal complements let us go the other way!

parametric $\rightarrow$ $\mathrm{Col}(A) = \mathrm{Nul}(A^T)^\perp$

$$\mathrm{Nul}(A^T) \xrightarrow{\text{PVF}} \mathrm{Span}\{v_1, \dots, v_{m-r}\}$$

$$\Rightarrow \mathrm{Col}(A) = \mathrm{Span}\{v_1, \dots, v_{m-r}\}^\perp$$

$$= \mathrm{Nul}\begin{pmatrix} — v_1^T — \\ \vdots \\ — v_{m-r}^T — \end{pmatrix} \leftarrow \text{implicit}$$

Null Space: implicit form $\quad \xrightarrow{\text{PVF}} \quad$ **Column Space:** parametric form

both require elimination

$\xleftarrow{(-)^\perp \text{ then PVF then } (-)^\perp}$

Like: easy to check if $x \in V$: $Ax = 0$

Like: can produce vectors in V: $x = a_1 v_1 + \cdots + a_n v_n$

Eg: Find an implicit equation for the plane

$$V = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\} \quad \text{parametric description}$$

$$V^\perp = \mathrm{Nul}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 0\end{pmatrix} \xrightarrow{\text{PVF}} \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}$$

$$\Rightarrow V = \mathrm{Span}\left\{\begin{pmatrix}-1\\1\\0\end{pmatrix}\right\}^\perp = \mathrm{Nul}\begin{pmatrix}-1 & 1 & 0\end{pmatrix}$$

$$= \left\{\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} : \underbrace{-x_1 + x_2 = 0}_{\text{implicit equation}}\right\}$$