Orthogonal Projections

Recall: to find the best approximate solution of $Ax=b$, want to find the closest vector $\hat{b}$ to b in $\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\}$

Want: $b - \hat{b}$ is orthogonal to V: $b - \hat{b} \in V^\perp$

[demo]

Def: Let V be a subspace of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$. The orthogonal projection of b onto V is the closest vector b_V in V to b. It is defined by

$$\boxed{b - b_V \in V^\perp.}$$

The orthogonal decomposition of b relative to V is

$$\boxed{b = b_V + b_{V^\perp}}$$

Here $b_{V^\perp} = b - b_V \in V^\perp$. Note that

$$b - b_{V^\perp} = b_V \in V = (V^\perp)^\perp$$

So that $b_{V^\perp}$ is projection onto $V^\perp$.

In other words, the orthogonal decomposition is

$$\boxed{\begin{aligned} b &= \begin{pmatrix}\text{closest vector } b_V \\ \text{to } b \text{ in } V\end{pmatrix} + \begin{pmatrix}\text{closest vector } b_{V^\perp} \\ \text{to } b \text{ in } V^\perp\end{pmatrix} \\ b &= \begin{pmatrix}\text{projection of } b \\ \text{onto } V\end{pmatrix} + \begin{pmatrix}\text{projection of } b \\ \text{onto } V^\perp\end{pmatrix}\end{aligned}}$$

[demos]

How to compute b_V?

Step 0: Write V as a column space or a null space.

$V = \mathrm{Col}(A)$: then $V^\perp = \mathrm{Nul}(A^T)$, so

$$b - b_V \in \mathrm{Nul}(A^T) \implies A^T(b - b_V) = 0$$

If $b_V \in \mathrm{Col}(A)$ then $b_V = A\hat{x}$ for $\hat{x} \in \mathbb{R}^n$:

$$A^T(b - A\hat{x}) = 0 \implies A^TA\hat{x} = A^Tb$$

Solve this equation for $\hat{x}$ $\rightsquigarrow$ $b_V = A\hat{x}$

Eg: Let $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $V = \mathrm{Col}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Find $b_V =$ the orthogonal projection of b to V.

We set up the equations $A^TA\hat{x} = A^Tb$:

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In augmented matrix form, $A^TA\hat{x} = A^Tb$ is:

$$\left(\begin{array}{cc|c} 3 & 2 & 1 \\ 2 & 2 & 1 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 1/2 \end{array}\right)$$

So $\hat{x} = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix} \Longrightarrow b_V = A\hat{x} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$

[demo]

Check: $b_{V\perp} = b - b_V = \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$

columns of A

$$\begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow b_{V\perp} \in \mathrm{Col}(A)^\perp \quad \checkmark$$

Distance from V: $\|b - b_V\| = \|b_{V\perp}\| = \left\|\begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}\right\| = \frac{1}{\sqrt{2}}$

Orthogonal Decomposition: $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1/2 \\ -1/2 \\ 0 \end{pmatrix}$

Procedure: To compute the orthogonal projection b_V of b onto $V = \mathrm{Col}(A)$:

(1) Solve the equation $A^TA\hat{x} = A^Tb$

(2) Then $b_V = A\hat{x}$ for any solution $\hat{x}$.

Then $b_{V^\perp} = b - b_V$, and the orthogonal decomposition of b relative to V is

$$b = b_V + b_{V^\perp}.$$

The distance from b to V is $\|b_{V^\perp}\|$.

Eg: Let $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $V = \mathrm{Col}\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ 1 & -1 & -1 \end{pmatrix}$.

Find the orthogonal decomposition of b relative to V.

(1) $A^TA = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ -1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ 1 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 6 \\ 0 & 3 & 6 \\ 6 & 6 & 18 \end{pmatrix}$

$A^Tb = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \\ -1 & 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$

Solve $A^TA\hat{x} = A^Tb$:

$$\left(\begin{array}{ccc|c} 6 & 0 & 6 & 4 \\ 0 & 3 & 6 & -1 \\ 6 & 6 & 18 & 2 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2/3 \\ 0 & 1 & 2 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

PVF $\rightsquigarrow$ $$\hat{x} = \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

(2) $b_V = A\hat{x}$ for any solution. Let's use the particular solution:

$$b_V = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 2/3 \\ -1/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

NB: $b_V = b$: what does that mean? b was already in V! More on this later.

Def: The normal equation of $Ax = b$ is
$$A^T A \hat{x} = A^T b$$

Fact: $A^T A \hat{x} = A^T b$ is always consistent! (Otherwise the Procedure wouldn't work.)

Why? I claim $\mathrm{Col}(A^T) = \mathrm{Col}(A^T A)$.

From before: $\mathrm{Nul}(A) = \mathrm{Nul}(A^T A)$

Take $(\cdot)^\perp$: $\mathrm{Nul}(A)^\perp = \mathrm{Nul}(A^T A)^\perp$

$$\mathrm{Nul}(A)^\perp = \mathrm{Row}(A) = \mathrm{Col}(A^T)$$
$$\mathrm{Nul}(A^T A)^\perp = \mathrm{Row}(A^T A) = \mathrm{Col}((A^T A)^T)$$
$$= \mathrm{Col}(A^T A) \checkmark$$

Since $A^Tb \in \mathrm{Col}(A^T) = \mathrm{Col}(A^TA)$, the equation $A^TA\hat{x} = A^Tb$ is consistent. ✓

NB: If $\hat{x}$ and $\hat{y}$ both solve

$$A^TA\hat{x} = A^Tx = A^TA\hat{y}$$

then $0 = A^TA\hat{x} - A^TA\hat{y} = A^TA(\hat{x}-\hat{y})$

$$\Rightarrow \hat{x}-\hat{y} \in \mathrm{Nul}(A^TA) \overset{\text{Fact}}{=} \mathrm{Nul}(A) \Rightarrow A(\hat{x}-\hat{y}) = 0$$

$\Rightarrow b_V = A\hat{x} = A\hat{y}$. So any soln of $A^TA\hat{x} = A^Tb$ works.

Now we know how to project onto a column space. What if $V = \mathrm{Nul}(A)$?

Then $V^\perp = \mathrm{Nul}(A)^\perp = \mathrm{Row}(A) = \mathrm{Col}(A^T)$.

So first compute $b_{V^\perp}$ = projection onto a col space, then $b_V = b - b_{V^\perp}$.

Procedure: To compute the orthogonal projection b_V of b onto $V = \mathrm{Nul}(A)$:

(1) Compute $b_{V^\perp}$ = projection onto $V^\perp = \mathrm{Col}(A^T)$ using the normal equation:

$$AA^T\hat{x} = Ab \rightsquigarrow b_{V^\perp} = A^T\hat{x}$$

(2) $b_V = b - b_{V^\perp}$

Eg: Project $b=\begin{pmatrix}1\\0\\0\end{pmatrix}$ onto $V=\mathrm{Nul}\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}$.

We need to solve $AA^T\hat{x}=Ab$

$$AA^T=\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix}=\begin{pmatrix}3&2\\2&2\end{pmatrix}$$

$$Ab=\begin{pmatrix}1&1&1\\1&1&0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$$

$$\rightsquigarrow \left(\begin{array}{cc|c}3&2&1\\2&2&1\end{array}\right)\xrightarrow{\text{rref}}\left(\begin{array}{cc|c}1&0&0\\0&1&1/2\end{array}\right)$$

So $\hat{x}=\begin{pmatrix}0\\1/2\end{pmatrix} \Longrightarrow b_{V^\perp}=A^T\hat{x}=\begin{pmatrix}1&1\\1&1\\1&0\end{pmatrix}\begin{pmatrix}0\\1/2\end{pmatrix}=\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}$

$$\Longrightarrow b_V=\begin{pmatrix}1\\0\\0\end{pmatrix}-\begin{pmatrix}1/2\\1/2\\0\end{pmatrix}=\begin{pmatrix}1/2\\-1/2\\0\end{pmatrix}$$

Projection onto a Line:

Suppose $V=\mathrm{Span}\{v\}$.

Then $V=\mathrm{Col}(A)$ where $A=v$ (one column).

$A^TA=v^Tv=v\cdot v$ is a scalar

$A^Tb=v^Tb=v\cdot b$

so the normal equation becomes

$$A^TA\hat{x}=A^Tb \rightsquigarrow (v\cdot v)\hat{x}=v\cdot b$$

Then $\hat{x} = \dfrac{v\cdot b}{v\cdot v} \rightsquigarrow b_V = A\hat{x} = \dfrac{v\cdot b}{v\cdot v}\, v$

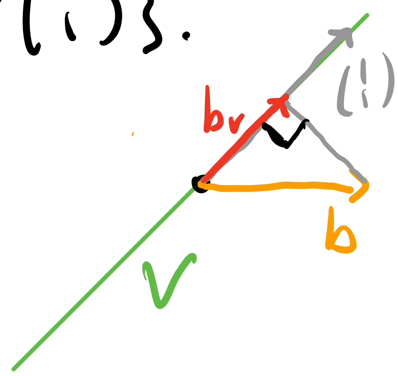
Here's the formula:

$$\boxed{\begin{array}{c}\text{Projection onto the Line Span}\{v\}\\[4pt] b_V = \dfrac{v\cdot b}{v\cdot v}\, v\end{array}}$$

Eg: Project $b = \begin{pmatrix}1\\0\end{pmatrix}$ onto $V = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$$b_V = \frac{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\end{pmatrix}}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}$$

[demo]



Properties of Projections: recall $b = b_V + b_{V^\perp}$

(1) $b_V = b \iff b_{V^\perp} = 0$

$\iff b \in V$

Think: the closest vector to b in V is b

$\iff$ b is already in V.

(2) $b_V = 0 \iff b = b_{V^\perp}$

$\iff b \in V^\perp$

(3) $(b_V)_V = b_V$ $\quad$ ($b_V \in V$; use (1))

Projection Matrices

Suppose that A has full column rank.
In this case, the $n \times n$ matrix A^TA is invertible:
this is HW 7 #13.

Fact: If $V = \mathrm{Col}(A)$ and A has full column rank then for any $b \in \mathbb{R}^m$,

$$\boxed{b_V = A(A^TA)^{-1}A^Tb}$$

This is because $A^TA\hat{x} = A^Tb$ has the unique solution $\hat{x} = (A^TA)^{-1}A^Tb$, so

$$b_V = A\hat{x} = A(A^TA)^{-1}A^Tb$$

Eg: $V = \mathrm{Col}(A)$ $\quad A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 2 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{6-4}\begin{pmatrix} 2 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 3/2 \end{pmatrix}$$

$$A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 3/2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2 \\ 0 & 1/2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So if $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then

$$b_V = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}$$

(cf. p.3)

Observation: $P_V = A(A^TA)^{-1}A^T$ is an $m\times m$ matrix that computes orthogonal projections onto $V = \mathrm{Col}(A)$: $P_V b = b_V$ for all $b \in \mathbb{R}^m$.

Def: Let V be a subspace of $\mathbb{R}^m$. The **projection matrix** for V is the $m\times m$ matrix P_V such that $P_V b = b_V$ for all $b \in \mathbb{R}^m$.

NB: The matrix P_V is *defined* by the equality

$$P_V b = b_V$$

for all vectors b. This uniquely characterizes P_V by the **Fact** below. Use the above equation to answer questions about P_V!

Fact: If A & B are $m\times n$ matrices and $Ax = Bx$ for all x, then $A = B$. Indeed, $Ae_i = i^{th}$ col of A.

What if $V = \mathrm{Col}(A)$ but A does not have full column rank?

Eg: $V = \mathrm{Col}(A)$ $\quad A = \begin{pmatrix} 1 & -1 & -1 \\ 2 & 1 & 4 \\ 1 & -1 & -1 \end{pmatrix}$

This A does not have full column rank:

$$A \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

This says that $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is a basis for V. This means:

(1) $V = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\} = \mathrm{Col}\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$

(2) $\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is LI

$\rightsquigarrow \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$ has full column rank.

So replace A by $B = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}$:

$$B^T B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 0 \\ 0 & 3 \end{pmatrix}$$

$$(B^T B)^{-1} = \begin{pmatrix} 1/6 & 0 \\ 0 & 1/3 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$P_V = B(B^TB)^{-1}B^T = \frac{1}{6}\begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & -2 \\ 2 & 2 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 3 & 0 & 3 \\ 0 & 6 & 0 \\ 3 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

So $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow b_V = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(cf. p.5) ✓

Procedure for Computing P_V:

(1) Find a basis $\{v_1, \ldots, v_n\}$ of V

for example, if $V = \mathrm{Col}(A)$ then use the pivot columns

(2) $B = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$

(3) $P_V = B(B^TB)^{-1}B^T$

Eg: Suppose $V = \mathrm{Span}\{v\}$ is a line.

$B = v$ (matrix with one column)

$B^TB = v \cdot v$ (a scalar)

$B(B^TB)^{-1}B^T = v(v\cdot v)^{-1}v^T = \dfrac{vv^T}{v\cdot v}$

outer product

Projection Matrix onto a Line

$$\boxed{\text{If } V = \text{Span}\{v\} \text{ then } P_V = \frac{vv^T}{v \cdot v}}$$

Eg: $V = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

$$P_V = \frac{1}{\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\end{pmatrix}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix} = \begin{pmatrix}1/2 & 1/2\\1/2 & 1/2\end{pmatrix}$$

So if $b = \begin{pmatrix}1\\0\end{pmatrix}$ then $b_V = P_V b = \begin{pmatrix}1/2\\1/2\end{pmatrix}$ (cf p.8) ✓

Properties of Projection Matrices:

Let V be a subspace of $\mathbb{R}^m$ and let P_V be its projection matrix.

$$\boxed{\begin{array}{ll} (1)\ \text{Col}(P_V) = V & (3)\ P_V^2 = P_V \\ (2)\ \text{Nul}(P_V) = V^\perp & (4)\ P_V + P_{V^\perp} = I_m \\ \multicolumn{2}{c}{(5)\ P_V = P_V^T} \end{array}}$$

Recall: A (square) matrix S is symmetric if $S = S^T$.

Proofs of the Properties:

This is a translation of properties of projections.

(1) $\mathrm{Col}(P_V) = \{P_V b : b \in \mathbb{R}^m\} = \{b_V : b \in \mathbb{R}^m\}$

This equals V

- because $b_V \in V$ for any b,
- and $b_V = b$ for any $b \in V$.

(2) $\mathrm{Nul}(P_V) = \{b \in \mathbb{R}^m : P_V b = 0\} = \{b \in \mathbb{R}^m : b_V = 0\}$

But we know $b_V = 0 \iff b \in V^\perp$.

(3) For any vector b,

$$P_V^2 b = P_V(P_V b) = P_V(b_V) = (b_V)_V$$

This equals b_V because $b_V \in V$ already

$$= b_V = P_V b$$

Since $P_V^2 b = P_V b$ for all vectors b, $P_V^2 = P_V$.

(4) For any vector b,

$$(P_V + P_{V^\perp})b = P_V b + P_{V^\perp} b = b_V + b_{V^\perp}$$

This equals b because $b = b_V + b_{V^\perp}$ is the orthogonal decomposition.

$$= b = I_m b$$

Since $(P_V + P_{V^\perp})b = I_m b$ for all vectors b,
$P_V + P_{V^\perp} = I_m$.

(5) Choose a basis for $V \rightsquigarrow P_V = B(B^TB)^{-1}B^T$.

$$P_V^T = \left(B(B^TB)^{-1}B^T\right)^T = B^{TT}\left((B^TB)^{-1}\right)^T B^T$$

$$= B\left((B^TB)^T\right)^{-1}B^T = B(B^TB)^{-1}B^T = P_V$$

For any invertible matrix A,

$$(A^{-1})^T = (A^T)^{-1} \quad \text{because}$$

$$(A^{-1})^T A^T = (AA^{-1})^T = I_n^T = I_n \quad \checkmark$$

Eg: Find P_V if $V = \mathrm{Nul}(A)$.
In this case, $V^\perp = \mathrm{Col}(A^T)$, so we know how to compute $P_{V^\perp}$. Then

$$P_V = I_n - P_{V^\perp}.$$

Eg: $V = \mathrm{Nul}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \rightsquigarrow V^\perp = \mathrm{Col}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

This is a line:

$$P_{V^\perp} = \frac{1}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

$$P_V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{pmatrix}$$

This was much easier than finding a basis for V using PVF, then using $P_V = A(A^TA)^{-1}A^T$.

$\rightarrow$ Be intelligent about what you actually have to compute!