Orthogonal Bases

Last time: we found the best approximate soln of $Ax=b$ using least squares.

Now we turn to computational considerations.

The goal is the QR decomposition.

LU makes solving $Ax=b$ fast	QR makes least-□ solving $Ax=b$ fast.

("fast" means: no elimination necessary)

The basic idea is that projections are easier when you have a basis of orthogonal vectors.

Def: A set of nonzero vectors $\{u_1, \ldots, u_n\}$ is:

(1) orthogonal if $u_i \cdot u_j = 0$ for $i \neq j$

(2) orthonormal if they're orthogonal and $u_i \cdot u_i = 1$ for all i (unit vectors).

Let $Q = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$, so $Q^TQ = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \cdots \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$.

(1) $\{u_1, \ldots, u_n\}$ is orthogonal $\Leftrightarrow$ Q^TQ is diagonal (& invertible)

all nonzero entries are on the diagonal

(2) $\{u_1, \ldots, u_n\}$ is orthonormal $\Leftrightarrow$ $Q^TQ = I_n$

Q: Does $Q^TQ = I_n$ mean $Q^T = Q^{-1}$?

$\to$ only if Q is square

Eg: $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$

(1) $u_1 \cdot u_2 = 0 \implies \{u_1, u_2\}$ is orthogonal

(2) $u_1 \cdot u_1 = 4 = u_2 \cdot u_2 \implies \{u_1, u_2\}$ is not orthonormal

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow Q^TQ = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

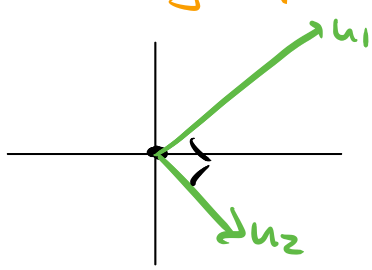
NB: Given an orthogonal set $\{u_1, \dots, u_n\}$ you can make it orthonormal by dividing by lengths:

$$v_i = \frac{u_i}{\|u_i\|} \rightsquigarrow \{v_1, \dots, v_n\} \text{ is orthonormal}$$

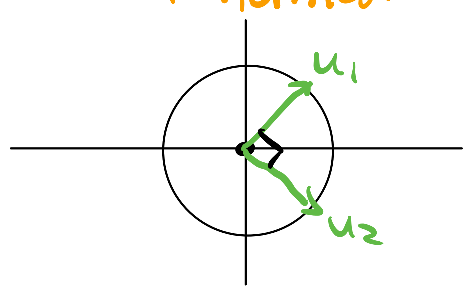
Eg: $v_1 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad v_2 = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$ $\quad \{v_1, v_2\}$ is o.n.

Picture in $\mathbb{R}^2$:

orthogonal



orthonormal



Fact:

Let $\{u_1, \dots, u_n\}$ be an orthogonal set and let $Q = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix}$. Then $\{u_1, \dots, u_n\}$ is linearly independent. Equivalently, Q has full column rank.

This means $\{u_1, \dots, u_n\}$ is a basis for $\mathrm{Span}\{u_1, \dots, u_n\}$

Proof: Say $x_1 u_1 + \cdots + x_n u_n = 0$. Take $(\cdot) \cdot u_1$:

$$0 = 0 \cdot u_1 = (x_1 u_1 + \cdots + x_n u_n) \cdot u_1$$
$$= x_1 u_1 \cdot u_1 + x_2 \cancel{u_2 \cdot u_1}^{0} + \cdots + x_n \cancel{u_n \cdot u_1}^{0}$$
$$= x_1 \|u_1\|^2 \implies x_1 = 0$$

Do the same for $u_2, u_3, \dots$ ✓

Projection formula:

Let $\{u_1, \dots, u_n\}$ be an orthogonal set and let $V = \mathrm{Span}\{u_1, \dots, u_n\}$. For any vector b,

$$\boxed{b_V = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} u_2 + \cdots + \frac{b \cdot u_n}{u_n \cdot u_n} u_n}$$

[demo]

NB: faster than $A^T A \hat{x} = A^T b$: no elimination necessary!

NB: $n=1 \rightsquigarrow$ get projection onto a line $b_V = \frac{b\cdot v}{v\cdot v}v$

Proof: Let $b' = \frac{b\cdot u_1}{u_1\cdot u_1}u_1 + \frac{b\cdot u_2}{u_2\cdot u_2}u_2 + \cdots + \frac{b\cdot u_n}{u_n\cdot u_n}u_n$.

We need $b - b' \in V^\perp$, ie $(b-b')\cdot u_i = 0$ for all i.

$$(b-b')\cdot u_1 = b\cdot u_1 - \left[\frac{b\cdot u_1}{\cancel{u_1\cdot u_1}}\cancel{u_1\cdot u_1} + \frac{b\cdot u_2}{u_2\cdot u_2}\overset{0}{\cancel{u_2\cdot u_1}} + \cdots + \frac{b\cdot u_n}{u_n\cdot u_n}\overset{0}{\cancel{u_n\cdot u_1}}\right]$$

$$= b\cdot u_1 - b\cdot u_1 = 0$$

Do the same for $u_2, u_3, \ldots$ ✓

Eg: Find the projection of $b = \begin{pmatrix}1\\2\\3\\2\end{pmatrix}$ onto

$$V = \operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\}$$

These vectors are orthogonal, so

$$b_V = \frac{\begin{pmatrix}1\\2\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{\begin{pmatrix}1\\2\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}}{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$$

$$= \frac{8}{4}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} + \frac{-2}{4}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix} = \begin{pmatrix}3/2\\3/2\\5/2\\5/2\end{pmatrix}$$

Projection Matrix: Outer Product Form

Let $\{u_1, \ldots, u_n\}$ be an orthogonal set and let $V = \text{Span}\{u_1, \ldots, u_n\}$. Then

$$\boxed{P_V = \frac{u_1 u_1^T}{u_1 \cdot u_1} + \frac{u_2 u_2^T}{u_2 \cdot u_2} + \cdots + \frac{u_n u_n^T}{u_n \cdot u_n}}$$

NB: outer product forms of matrices will be a key part of the SVD.

Proof:
$$\left(\frac{u_1 u_1^T}{u_1 \cdot u_1} + \frac{u_2 u_2^T}{u_2 \cdot u_2} + \cdots + \frac{u_n u_n^T}{u_n \cdot u_n}\right) b$$

This is the defining property of P_V

$$= \frac{u_1}{u_1 \cdot u_1}(u_1^T b) + \cdots + \frac{u_n}{u_n \cdot u_n}(u_n^T b)$$

$$= \frac{u_1 \cdot b}{u_1 \cdot u_1} u_1 + \cdots + \frac{u_n \cdot b}{u_n \cdot u_n} u_n = b_V = P_V b$$

Eg: Find P_V for $V = \text{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\}$

$$P_V = \frac{1}{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\\1\end{pmatrix}} \begin{pmatrix}1\\1\\1\\1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\end{pmatrix} + \frac{1}{\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}} \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\begin{pmatrix}1 & 1 & -1 & -1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\\1 & 1 & 1 & 1\end{pmatrix} + \frac{1}{4}\begin{pmatrix}1 & 1 & -1 & -1\\1 & 1 & -1 & -1\\-1 & -1 & 1 & 1\\-1 & -1 & 1 & 1\end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix}1 & 1 & 0 & 0\\1 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{pmatrix}$$

Now we consider orthonormal vectors.

Facts:

Let $\{v_1, \ldots, v_n\}$ be an orthonormal set and let $Q = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$.

(1) $Q^T Q = I_n$

(2) $(Qx)\cdot(Qy) = x \cdot y$ for all $x, y \in \mathbb{R}^n$

(3) $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$

(4) Let $V = \text{Span}\{v_1, \ldots, v_n\} = \text{Col}(Q)$. Then

$$\boxed{P_V = QQ^T}$$

NB: (2) says $(Q\cdot)$ does not change angles.
(3) says $(Q\cdot)$ does not change lengths.

Proofs: (1) cf. p. 1 ✓

(2) $(Qx)\cdot(Qy) = (Qx)^T Q y = x^T Q^T Q y = x^T I_n y$
$= x \cdot y$ ✓

(3) $\|Qx\| = \sqrt{(Qx)\cdot(Qx)} \overset{(2)}{=} \sqrt{x \cdot x} = \|x\|$ ✓

(4) $P_V = Q(Q^T Q)^{-1} Q^T = Q(I_n)^{-1} Q^T = QQ^T$ ✓

Eg: Find P_V for $V = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\}$

This has an orthonormal basis $\frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \frac{1}{2}\begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}$

$$Q = \frac{1}{2}\begin{pmatrix}1 & 1\\1 & 1\\1 & -1\\1 & -1\end{pmatrix}$$

$$P_V = QQ^T = \frac{1}{4}\begin{pmatrix}1 & 1\\1 & 1\\1 & -1\\1 & -1\end{pmatrix}\begin{pmatrix}1 & 1 & 1 & 1\\1 & 1 & -1 & -1\end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix}2 & 2 & 0 & 0\\2 & 2 & 0 & 0\\0 & 0 & 2 & 2\\0 & 0 & 2 & 2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & 1 & 0 & 0\\1 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{pmatrix}$$

The projection formula is easier with no denominators:

Projection formula for an Orthonormal Basis:

Let $\{v_1, \ldots, v_n\}$ be an orthonormal set and let $V = \mathrm{Span}\{v_1, \ldots, v_n\}$. For any vector b,

$$\boxed{b_V = (b \cdot v_1)v_1 + (b \cdot v_2)v_2 + \cdots + (b \cdot v_n)v_n}$$

Moreover,

$$\boxed{P_V = v_1 v_1^T + v_2 v_2^T + \cdots + v_n v_n^T}$$

Def: A square matrix with orthonormal columns is called orthogonal.

↳ Note the strange terminology!

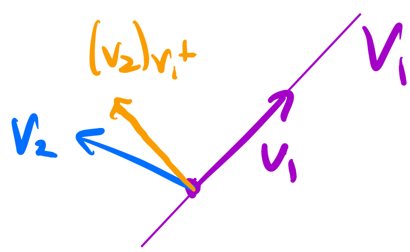
Q: Why is $P_V = QQ^T = I_n$?

Gram-Schmidt

Given that projections are easier to compute in an orthogonal basis, how do we produce one?

Idea: Start with any basis $\{v_1, \dots, v_n\}$

- make $v_2 \perp v_1$ by replacing with $(v_2)_{V_1^\perp}$ for $V_1 = \mathrm{Span}\{v_1\}$



- make $v_3 \perp v_1, v_2$ by replacing with $(v_3)_{V_2^\perp}$ for $V_2 = \mathrm{Span}\{v_1, v_2\}$
- etc

This "straightens out" the basis vectors one at a time.

NB: $(v_3)_{V_2^\perp}$ is easy to compute w/ projection formula!

Procedure (Gram–Schmidt):

Let $\{v_1, \dots, v_n\}$ be a basis for a subspace V.

(1) $u_1 := v_1$

(2) $u_2 = v_2 - \dfrac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$ $\quad = (v_2)_{V_1^\perp} \quad V_1 = \text{Span}\{v_1\}$

(3) $u_3 = v_3 - \dfrac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \dfrac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2$ $\quad = (v_3)_{V_2^\perp} \quad V_2 = \text{Span}\{v_1, v_2\}$

$\vdots$

(n) $u_n = v_n - \dfrac{u_1 \cdot v_n}{u_1 \cdot u_1} u_1 - \dfrac{u_2 \cdot v_n}{u_2 \cdot u_2} u_2 - \cdots - \dfrac{u_{n-1} \cdot v_n}{u_{n-1} \cdot u_{n-1}} u_{n-1}$

Then $\{u_1, \dots, u_n\}$ is an orthogonal basis for V, and

$$\text{Span}\{u_1, \dots, u_i\} = \text{Span}\{v_1, \dots, v_i\} \text{ for } 1 \le i \le n$$

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$$u_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$u_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

output: $u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ $\quad u_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

check: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$ $\quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ $\quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ ✓

Q: What if $\{v_1, \dots, v_n\}$ is linearly dependent?

Then eventually $v_i \in \mathrm{Span}\{v_1, \dots, v_{i-1}\} = \mathrm{Span}\{u_1, \dots, u_{i-1}\}$

so $v_i \in V_{i-1} = \mathrm{Span}\{u_1, \dots, u_{i-1}\} \implies u_i = (v_i)_{V_{i-1}^\perp} = 0$

This is ok! Just discard v_i & continue.

QR Decomposition

This "keeps track" of the Gram-Schmidt procedure in the same way that LU keeps track of row operations.

Start with a basis $\{v_1, \dots, v_n\}$ of a subspace & run Gram-Schmidt. Then

Solve for v_i's in terms of u_i's:

$$v_1 = u_1$$
$$v_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 + u_2$$
$$v_3 = \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 + u_3$$
$$v_4 = \frac{v_4 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_4 \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{v_4 \cdot u_3}{u_3 \cdot u_3} u_3 + u_4$$

Matrix Form:

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} 1 & \frac{v_2 \cdot u_1}{u_1 \cdot u_1} & \frac{v_3 \cdot u_1}{u_1 \cdot u_1} & \frac{v_4 \cdot u_1}{u_1 \cdot u_1} \\ 0 & 1 & \frac{v_3 \cdot u_2}{u_2 \cdot u_2} & \frac{v_4 \cdot u_2}{u_2 \cdot u_2} \\ 0 & 0 & 1 & \frac{v_4 \cdot u_3}{u_3 \cdot u_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

QR Decomposition:

Let A be an $m \times n$ matrix with full column rank. Then

$$A = QR$$

where

- Q is an $m \times n$ matrix whose colums form an orthonormal basis of $\mathrm{Col}(A)$
- R is upper-Δ $n \times n$ with nonzero diagonal entries.

To compute Q & R: let $\{v_1, \dots, v_n\}$ be the columns of A. Run Gram-Schmidt w/ $\{u_1, \dots, u_n\}$. Then

cancels

$$Q = \begin{pmatrix} | & | & | & | \\ \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} & \frac{u_4}{\|u_4\|} \\ | & | & | & | \end{pmatrix} \qquad R = \begin{pmatrix} \|u_1\| & \frac{v_2 \cdot u_1}{u_1 \cdot u_1}\|u_1\| & \frac{v_3 \cdot u_1}{u_1 \cdot u_1}\|u_1\| & \frac{v_4 \cdot u_1}{u_1 \cdot u_1}\|u_1\| \\ 0 & \|u_2\| & \frac{v_3 \cdot u_2}{u_2 \cdot u_2}\|u_2\| & \frac{v_4 \cdot u_2}{u_2 \cdot u_2}\|u_2\| \\ 0 & 0 & \|u_3\| & \frac{v_4 \cdot u_3}{u_3 \cdot u_3}\|u_3\| \\ 0 & 0 & 0 & \|u_4\| \end{pmatrix}$$

Analogy to LU decomposition:

$$A = L\,U \qquad\qquad A = Q\,R$$

L: steps to get to echelon form

U: echelon form

Q: o.n. basis

R: steps to get to o.n. basis

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ $\quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 6 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\times\sqrt{2} = \|u_1\|$

$$u_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{-6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$\times\sqrt{2} = \|u_1\|$

$$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\times\sqrt{6} = \|u_2\|$

$$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 3\sqrt{2} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

Application: makes least-□ faster!

Given $A = QR$, to solve $Ax = b$ by least-□:

$$A^TA\hat{x} = (QR)^T(QR)\hat{x} = R^TQ^TQR\hat{x} = R^TI_nR\hat{x} = R^TR\hat{x}$$

$$A^Tb = (QR)^Tb = R^TQ^Tb$$

NB: R is invertible: it is upper-△ with nonzero diagonal entries.

Solve $R^TR\hat{x} = R^TQ^Tb$: $(R^T)^{-1} \cdot (\cdot)$

$$\boxed{A^TA\hat{x} = A^Tb \iff R\hat{x} = Q^Tb}$$

R is upper-△: solve with back substitution!

NB: Can compute QR in $\sim \frac{10}{3}n^3$ flops for $n \times n$.
(not with this algorithm) Then need $O(n^2)$ flops to do least-$\square$ on $Ax=b$. (Multiply by Q^T & forward-substitute.) Much faster than $O(n^3)$!

Eg: Find the least squares soln of $Ax=b$ for

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 0 & -2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{using} \quad A = QR$$

$$\text{for} \quad Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{pmatrix} \quad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} \end{pmatrix}$$

$$Q^Tb = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4/\sqrt{6} \end{pmatrix}$$

$$R\hat{x} = Q^Tb \rightsquigarrow \left(\begin{array}{cc|c} \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{6} & 4/\sqrt{6} \end{array}\right) \rightsquigarrow \begin{array}{r} x_1\sqrt{2} + x_2\sqrt{2} = 0 \\ x_2\sqrt{6} = \frac{4}{\sqrt{6}} \end{array}$$

$$\Rightarrow x_2 = \frac{4}{6} = \frac{2}{3}, \quad x_1\sqrt{2} + \frac{2}{3}\sqrt{2} = 0$$

$$\Rightarrow x_1 = -\frac{2}{3} \quad \Rightarrow \hat{x} = \begin{pmatrix} 2/3 \\ -2/3 \end{pmatrix}$$