

Determinants

What we've done:

- Solve $Ax=b$
(Gauss-Jordan, LU, PVE, ...)
- Approximately solve $Ax=b$
(orthogonality, projections, QR, ...)

What's next:

- Solve $Ax=\lambda x$

This is the **eigenvalue problem** used in difference equations (rabbit population) & ODEs. It deals exclusively with **square matrices**.

The **determinant** of a **square** matrix is a number that satisfies many **magical properties**. I'll define it by telling you how to compute it using **row operations**.

→ Next time: other ways to compute it.

Def: The determinant of a square matrix A is a number $\det(A)$ or $|A|$ satisfying:

(1) If $A \xrightarrow{R_i + cR_j} B$ then $\det(A) = \det(B)$.

(2) If $A \xrightarrow{R_i \times c} B$ then $\det(A) = \frac{1}{c} \det(B)$.

(3) If $A \xrightarrow{R_i \leftrightarrow R_j} B$ then $\det(A) = -\det(B)$

(4) $\det(I_n) = 1$.

Consequence: if A has a zero row then $\det(A) = 0$

Eg: $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{(2)} -\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

$$\Rightarrow \det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Consequence: if A is (upper/lower) triangular then $\det(A) = \text{product of diagonal entries}$

$\det \begin{pmatrix} \text{triangular} \\ \text{matrix} \end{pmatrix} = \text{product of the diagonal entries}$

↘ eg. REF

Eg: $\det \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\substack{R_1 \times = \frac{1}{a} \quad R_2 \times = \frac{1}{b} \\ R_3 \times = \frac{1}{c}}]{(2)} abc \det \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$

$\xrightarrow[\text{replacements}]{\substack{\text{row (1)} \\ (4)}} abc \det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = abc$

What if $b=0$ though? (or $a?$ or $c?$)

$\det \begin{pmatrix} a & * & * \\ 0 & 0 & * \\ 0 & 0 & c \end{pmatrix} \xrightarrow[\text{replacements}]{\substack{\text{row (1)} \\ (4)}} \det \begin{pmatrix} a & * & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}$

$= 0 = a \cdot 0 \cdot c \quad \checkmark$

A REF matrix is triangular, so you can compute $\det(A)$ by Gaussian elimination!

Eg: $\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow{(3)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{pmatrix}$

$\xrightarrow{(1)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{(1)} -\det \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$

$= -2$

NB: You get the same number for $\det(A)$ no matter which row operations you do!

$$\text{Eg: } \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix} \xrightarrow[(3)]{R_1 \leftrightarrow R_3} -\det \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow[(1)]{R_2 - R_1} -\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow[(1)]{R_3 - R_2} -\det \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = -2 \checkmark$$

Gaussian elimination is the fastest general algorithm for computing the determinant of a matrix (with known entries).

Procedure: To compute $\det(A)$, run Gaussian elimination: $A \xrightarrow[\text{operations}]{\text{row}} U$. Then

$$\det(A) = (-1)^{\# \text{ row swaps}} \cdot \frac{1}{\prod (\text{row scaling})} \prod (\text{diagonal entries of } U)$$

NB: You don't need to do row scaling operations to run Gaussian elimination, so this term usually does not appear.

Eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

• If $a \neq 0$:

$$\det(A) \xrightarrow{R_2 \leftarrow \frac{c}{a}R_1} \det \begin{pmatrix} a & b \\ 0 & d - \frac{c}{a}b \end{pmatrix} = a(d - \frac{c}{a}b) = ad - bc$$

• If $a = 0$:

$$\det(A) \xrightarrow{R_1 \leftrightarrow R_2} -\det \begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = -bc = ad - bc$$

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

NB: If U is a REF of A then

$$\det(U) = \prod \text{diagonal entries}$$

$$\det(U) \neq 0 \iff \text{all diagonal entries are nonzero}$$

$$\iff U \text{ has } n \text{ pivots}$$

$$\iff A \text{ has } n \text{ pivots}$$

$$\iff A \text{ is invertible}$$

Note $\det(A) = (\text{nonzero scalar}) \cdot \det(U)$, so

$$A \text{ is invertible} \iff \det(A) \neq 0$$

Magical Properties of the Determinant:

(1) **Existence:** There exists a number $\det(A)$ satisfying defining properties (1) - (4).

(2) **Invertibility:** A is invertible $\iff \det(A) \neq 0$

(3) **Multiplicativity:** $\det(AB) = \det(A)\det(B)$
and $\det(A) \neq 0 \implies \det(A^{-1}) = \frac{1}{\det(A)}$

(4) **Transposes:** $\det(A^T) = \det(A)$

|| We can think of $\det(-)$ as a function of n vectors (the cols of A):
$$\det \begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} \longleftrightarrow \det(v_1, \dots, v_n)$$

(5) **Multilinearity:**

$$\begin{aligned} \det(v_1, \dots, av + bw, \dots, v_n) \\ = a \det(v_1, \dots, v, \dots, v_n) + b \det(v_1, \dots, w, \dots, v_n). \end{aligned}$$

NB: (1) just says: You get the same number for $\det(A)$ no matter which row ops you do!

Eg: $\det \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \xrightarrow[R_3 \leftarrow 7R_1]{R_2 \leftarrow 4R_1} \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix}$

$\xrightarrow{R_3 \leftarrow 2R_2} \det \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix} = 0$ ● = pivots

so $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is singular (not invertible)

NB: If the columns of A are linearly dependent then A does not have full column rank \Rightarrow not invertible $\Rightarrow \det(A) = 0$. Likewise for rows (take transposes).

A has linearly dependent rows or columns $\Rightarrow \det(A) = 0$

Eg: $\det \left[\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{100} \right]$

$= \det \left[\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{99} \right]$

$= \det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \det \left[\begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{99} \right]$

$= \dots = \left[\det \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right]^{100} = (-2)^{100}$

More generally,

$$\det(A^n) = \det(A)^n \quad \text{for all } n \geq 0 \\ \text{(and } n < 0 \text{ if } \det(A) \neq 0)$$

Eg: Say A has a $PA = LU$ decomposition.

$$\det(L) = \det \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix} = 1$$

You get P by doing row swaps on I_n , so

$$\det(P) = (-1)^{\# \text{row swaps}}$$

Hence

$$(-1)^{\# \text{row swaps}} \det(A) = \det(PA)$$

$$= \det(LU) = \det(L) \det(U)$$

$$= \det(U)$$

This recovers the formula on p.4 (we did no row scaling operations).

The transpose property says that $\det(A)$ satisfies (1)-(3) for **column operations** too: they're just row operations on A^T .

Eg: $\det \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{C_1 - 4C_3}}} \det \begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$

\parallel transpose \parallel transpose

$\det \begin{pmatrix} 2 & 3 & 4 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{R_1 - 4R_3}}} \det \begin{pmatrix} -14 & -9 & 0 \\ 7 & 1 & 0 \\ 4 & 3 & 1 \end{pmatrix}$

So we can compute \det using column ops:

$$\det \begin{pmatrix} 2 & 7 & 4 \\ 3 & 1 & 3 \\ 4 & 0 & 1 \end{pmatrix} \xrightarrow{\underline{\underline{C_1 - 4C_3}}} \det \begin{pmatrix} -14 & 7 & 4 \\ -9 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

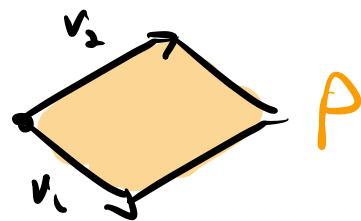
$$\xrightarrow{\underline{\underline{C_1 + 9C_2}}} \det \begin{pmatrix} 49 & 7 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 49$$

Determinants and Volumes

Where do properties (1) - (4) come from?

Two vectors $v_1, v_2 \in \mathbb{R}^2$ determine ("span") a **parallelogram**:



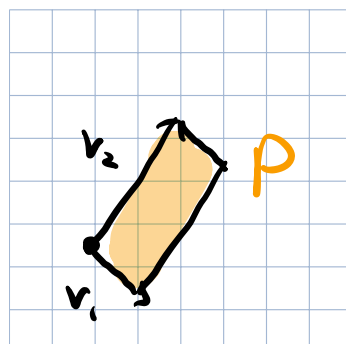
$$P = \{x_1 v_1 + x_2 v_2 : x_1, x_2 \in [0, 1]\}$$

Fact: $\text{area}(P) = \left| \det \begin{pmatrix} -v_1^T & - \\ -v_2^T & - \end{pmatrix} \right|$

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

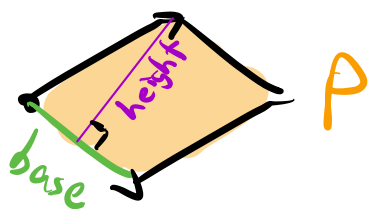
$$\text{area}(P) = \left| \det \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \right|$$

$$= |(3)(1) - (2)(-1)| = 5$$



Why? Let's check that $\text{area}(P)$ satisfies the four defining properties (1) - (4) of the determinant.

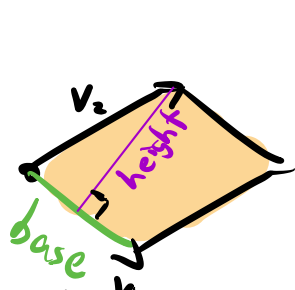
NB: $\text{area}(P) = \text{base} \times \text{height}$:



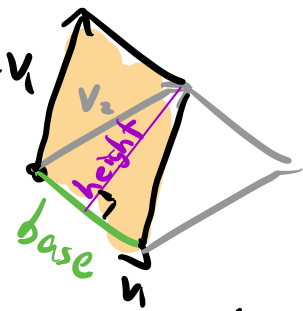
(1) Row replacement

$$v_2 \rightsquigarrow v_2 + cv_1$$

area = base \times ht : unchanged



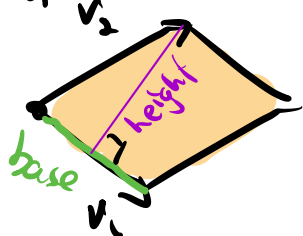
$$c = -\frac{2}{3}$$



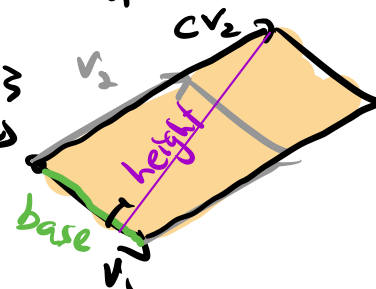
(2) Row scaling

$$v_2 \rightsquigarrow cv_2$$

ht scaled by $|c| \Rightarrow$ base \times ht : scaled by $|c|$



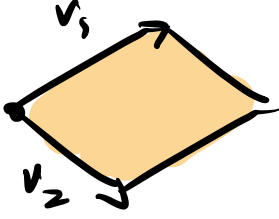
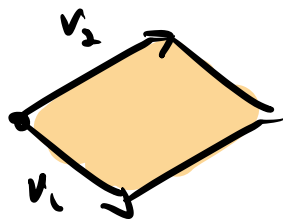
$$c = 1.3$$



(3) Row Swap

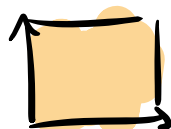
$$v_1 \leftrightarrow v_2$$

area unchanged = $|\det|$



(4) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

area = 1.

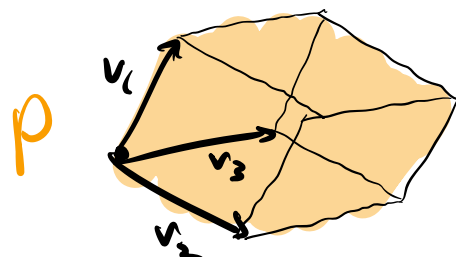


Q: minus sign?
HW 9

This generalizes as follows (same reasoning):


Def: The **paralleliped** determined ("spanned") by n vectors

$$P = \{x_1 v_1 + \dots + x_n v_n : x_1, \dots, x_n \in [0, 1]\}$$



Thm (Determinants & Volumes):

$$\text{volume}(P) = \left| \det \begin{pmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{pmatrix} \right|$$

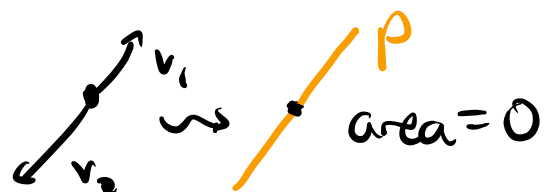
NB: When $n=1$, "volume" = "length":
 $\text{length}(a) = |a|$ 

NB: When $n=2$, "volume" = "area".

Question: When is $\text{volume}(P) = 0$?

When P is **squashed flat**:

ie when v_1, \dots, v_n are



linearly dependent ($\Rightarrow \det(\dots) = 0$)

NB: In multivariable calc, you approximate shapes by tiny cubes, which turn into tiny parallelepipeds after applying a function. This is why determinants appear in the **change of variables** formula for integrals.

if $(y_1, \dots, y_n) = \phi(x_1, \dots, x_n)$ then

$$dy_1 \cdots dy_n = \left| \det \left(\partial y_i / \partial x_j \right) \right| dx_1 \cdots dx_n$$