

## Math 218D Problem Session

Week 2

### 1. Row Echelon Form

For each of the following linear systems/augmented matrices, do the following:

- (1) If it is a linear system, convert it to an augmented matrix. If it is an augmented matrix, convert it to a linear system.
- (2) Decide whether or not the augmented matrix is in Row Echelon Form (REF). If it is in REF, circle the pivots/pivot entries. If it is not, explain why not.

a)

$$\begin{aligned}x + 2y &= 1 \\ 3y &= 2\end{aligned}$$

b)

$$\left( \begin{array}{cc|c} 5 & -1 & 1 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{array} \right)$$

c)

$$\begin{aligned}2x + y - z &= 1 \\ 3y + z &= 2 \\ 2x &= 1\end{aligned}$$

d)

$$\left( \begin{array}{cc|c} 2 & 1 & 3 \\ 0 & 1 & 5 \end{array} \right)$$

e)

$$\left( \begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -2 & 4 & 1 \end{array} \right)$$

f)

$$\left( \begin{array}{cc|c} 5 & 5 & 5 \\ 1 & 1 & 1 \end{array} \right)$$

## 2. Two Equations and Two Unknowns

Consider the system of 2 linear equations:

$$\begin{aligned}x - y &= 2 \\ 2x - 4y &= -4.\end{aligned}$$

a) Draw the two lines in  $\mathbf{R}^2$  determined by these two equations.

Now, solve the linear system using the following steps:

b) Use one row operation to eliminate  $x$  from the second equation.

c) Use another row operation to make the second equation into  $y = (?)$ .

d) Use a third row operation to make the first equation into  $x = (?)$ . What is the solution  $(x, y)$ ?

e) Check your answer by plugging your solution in to the original equations.

The first row operation is an *elimination* step, while the third is a *substitution* step.

f) After which row operation is the system in REF? Circle the pivot entries.

### 3. Three Equations Three Unknowns

Consider the system of three linear equations

$$\begin{aligned}x_1 - 3x_2 + x_3 &= 4 \\2x_1 - 8x_2 + 8x_3 &= -2 \\-6x_1 + 3x_2 - 15x_3 &= 9.\end{aligned}$$

a) Convert this linear system into a **matrix equation**  $Ax = b$ , where

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

b) Write down the augmented matrix  $(A \mid b)$ .

c) Use elementary row operations to convert the augmented matrix into a Row Echelon Form matrix.

**Hint:** Begin by eliminating  $x_1$ : add a multiple of the first row to the second and third rows, so that the 2 and  $-6$  in the first column are replaced with 0. This takes two row operations.

d) How many elementary row operations did you use?

e) Convert the augmented matrix back into a system of three linear equations, and use back-substitution to find the solution vector  $x$ .

f) Check your answer by multiplying  $A$  by  $x$  and confirming that it equals  $b$ .

### 4. Another One—What's Different?

Consider the system of three linear equations

$$\begin{aligned}x_1 - 2x_2 + x_3 &= -2 \\2x_1 - 4x_2 + 8x_3 &= 2 \\x_1 - 3x_2 - x_3 &= 1.\end{aligned}$$

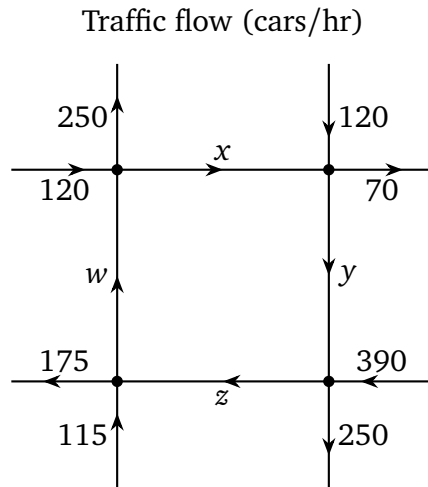
a) Use row operations to eliminate  $x_1$  from the second and third equation.

b) You can now use a single row operation to put the linear system in to REF. What row operation is it?

c) Substitute and solve for  $(x_1, x_2, x_3)$ . Check your answer with original linear system.

## 5. Traffic Jam

You should have seen this traffic example in lecture, although the numbers were a bit different. We'll now will explore it in more detail.



This represents a town with 4 main roads, as well as 8 roads in and out of town. Each road is one-way, in the direction indicated by the arrows. The 8 roads have a fixed number of cars/hour which travel in and out of town on them.

**Question:** How many cars/hour travel on each of the 4 main roads?

We'll use linear algebra to answer this question. At each intersection, the number of incoming cars per hour must equal the number of outgoing cars. This gives 4 linear equations, in the variables  $x, y, z, w$ :

$$120 + w = 250 + x$$

$$120 + x = 70 + y$$

$$390 + y = 250 + z$$

$$115 + z = 175 + w.$$

- a) Rewrite these equations with the variables on the left and the numbers on the right. Write the variables on the left side in neat columns.
- b) Use the first equation to eliminate  $x$  from the equations below it.
- c) Use the second equation to eliminate  $y$  from the equations below it.
- d) Use the third equation to eliminate  $z$  from the equations below it.

Remember: after each step, you should have a linear system with 4 equations and 4 variables.

- e) At this point you are done with elimination. Can you substitute to solve? Can you answer the original **Question**?
- f) How many solutions does this system have? Can you explain why from the original picture?

- g) Convert the linear system obtained after step 3 into an augmented matrix. It should be in REF. Circle the pivot entries. Does each row have a pivot entry? Does each column to the left of the augmentation line have a pivot entry?

## 6. Reduced Row Echelon Form

For each of the following augmented matrices:

a)  $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{array}\right),$

b)  $\left(\begin{array}{ccc|c} 1 & 0 & 3 & 2 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right),$

c)  $\left(\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{array}\right),$

d)  $\left(\begin{array}{c|c} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{array}\right),$

answer the following questions:

- (1) determine whether or not it is in REF;
- (2) determine whether or not it is in RREF;
- (3) if it is not in RREF, use the Gauss–Jordan algorithm to convert it into RREF;
- (4) circle the pivots of the augmented matrices and determine the number of solutions to the corresponding system of linear equations.