

Math 218D Problem Session

Week 10

1. Some simple examples

For each of the following matrices A ,

- i) Find the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.
- ii) Find all the *eigenvalues* by solving $p(\lambda) = 0$.
- iii) For each eigenvalue λ_i , find a basis of the associated *eigenspace* $\text{Nul}(A - \lambda_i I_2)$.
- iv) An $n \times n$ matrix A is diagonalizable if and only if the dimensions of the eigenspaces add up to n . For these matrices, you may have one or two eigenspaces, depending on how many different roots $p(\lambda)$ has.

Is the matrix A diagonalizable? Is the matrix A diagonal?

$$\text{a) } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{b) } \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{c) } \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{d) } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{e) } \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{f) } \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{g) } \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

2. A 2×2 diagonalization

Consider the matrix $A = \begin{pmatrix} -1 & 2 \\ -3 & 4 \end{pmatrix}$.

- a) Compute the characteristic polynomial $p(\lambda) = \det(A - \lambda I_2)$.
- b) Using the quadratic formula, find the two solutions to $p(\lambda) = 0$. The two solutions, λ_1 and λ_2 , are the two eigenvalues of A .
- c) Find the eigenvector $v_1 = (x_1, y_1)$ by solving the eigenvector equation

$$(A - \lambda_1 I_2)v_1 = 0$$

Note that there is more than one solution—choose any non-zero solution.

- d) Find the eigenvector $v_2 = (x_2, y_2)$ by solving the eigenvector equation

$$(A - \lambda_2 I_2)v_2 = 0.$$

- e) Diagonalize A , by making a matrix of eigenvalues $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$, a matrix of eigenvectors $C = \begin{pmatrix} v_1 & v_2 \end{pmatrix}$, and confirming that $A = CDC^{-1}$ by multiplying these three matrices.

- f) Compute the vector $A^n(1, 2)$.

Hint: Find scalars c_1, c_2 so that $(1, 2) = c_1 v_1 + c_2 v_2$. It may help to use the matrix C^{-1} to do this. Then use the formula $A^n(c_1 v_1 + c_2 v_2) = c_1 A^n v_1 + c_2 A^n v_2$.

- g) When n is very large, $\|A^{n+1}(1, 2)\|/\|A^n(1, 2)\|$ is approximately ____.
- h) When n is very large, $\|A^{n+1}(1, 1)\|/\|A^n(1, 1)\|$ is approximately ____ (this should be easier than g).)
- i) If you were given a random vector w , what would you expect $\|A^{n+1}w\|/\|A^n w\|$ to approximate when n is very large?

3. Some 3×3 characteristic polynomials

Compute the characteristic polynomials and eigenvalues of the matrices

$$A = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Decide if each matrix is diagonalizable, and if it is, diagonalize it.

4. Traces and determinants

Recall that the trace $\text{Tr}(A)$ is the sum of the diagonal entries of A .

- a) For each of the matrices in problem 1(a)–(f), factor $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)$. Verify that

$$\text{Tr}(A) = \lambda_1 + \lambda_2 \text{ and } \det(A) = \lambda_1 \cdot \lambda_2.$$

- b) For any $n \times n$ matrix, the polynomial $p(\lambda) = \det(A - \lambda I_n)$ can be factored as

$$p(\lambda) = (-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

Verify that

$$\det(A) = \lambda_1 \cdots \lambda_n.$$

Hint: What happens to $\det(A - \lambda I_n)$ when you set $\lambda = 0$? What happens to $(-1)^n(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ when you set $\lambda = 0$?

- c) The determinant $\det(A)$ has another product formula:

$$\det(A) = (-1)^k d_1 \cdots d_n,$$

when the A has REF with pivot entries d_1, \dots, d_n , found using Gaussian elimination w/o row scaling and with k row swaps. Even though this formula looks quite similar to the formula of **b**), eigenvalues and pivots are not at all the same.

Find an example of a 2×2 matrix where the pivots d_1, d_2 are not the same as the eigenvalues λ_1, λ_2 .

- d) **(Challenge)** For any $n \times n$ matrix, show that $\text{Tr}(A) = \lambda_1 + \cdots + \lambda_n$.

5. Linear independence of eigenvectors

- a) Consider a matrix A with two distinct eigenvalues $\lambda_1 \neq \lambda_2$, with associated eigenvectors v_1 and v_2 . Show that v_1 is not a scalar multiple of v_2 .

Hint: Suppose they were scalar multiples, $v_1 = cv_2$. What happens when you multiply this equation by A ?

- b) Consider a matrix A with three distinct eigenvalues $\lambda_1, \lambda_2, \lambda_3$, with associated eigenvectors v_1, v_2 and v_3 . Show that v_1, v_2 , and v_3 are linearly independent.

Hint: Suppose they were dependent, $a_1v_1 + a_2v_2 + a_3v_3 = 0$, with $a_3 \neq 0$. Multiply this equation by A . Can you get a new linear dependence where $a_3 = 0$?