Math 218D Problem Session

Week 11

1. Matrices with complex eigenvalues

Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) Compute the eigenvalues of A and B. Write each eigenvalue in polar coordinates $z = re^{i\theta}$.

b) Compute the eigenvectors of A and B.

2. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

a) For each of the following vectors, plot v, Av, and A^2v:

(1) $v = (1,0)$

(2) $v = (0,1)$

(3) $v = (1,1)$

b) For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v,\ A^{-1}v,\ v,\ Av,\ A^2v, \ldots$.

c) For the vector $v = (1,1)$, what direction is the vector $A^n v$ approximately pointing when n is very large? In other words, what unit vector does $\frac{A^n v}{||A^n v||}$ approximate when n is very large?

d) For the vector $v = (1,1)$, what direction is $A^{-n}v$ approximately pointing when n is very large?

3. The dynamics of a diagonalizable matrix

Consider the matrix A with $A(1,1) = 3(1,1)$ and $A(1,-2) = 2(1,-2)$. In other words, A is diagonalizable and you have been told the eigenvectors and eigenvalues.

a) For each of the following vectors, plot v, Av, A^2v:

(1) $v = (1,1)$

(2) $v = (1,-2)$

(3) $v = (2,-1)$

You can do this without computing the matrix A!

b) For each of the same vectors, sketch the shape you get by connecting the dots between the points $\ldots, A^{-2}v,\ A^{-1}v,\ v,\ Av,\ A^2v, \ldots$.

c) For the vector $v = (2,-1)$, what direction is the vector $A^n v$ approximately pointing when n is very large?

d) For the vector $v = (2,-1)$, what direction is $A^{-n}v$ approximately pointing when n is very large?

4. Dynamics of complex matrices

Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ you studied in problem 1.

a) Plot the points $(4,0)$, $A(4,0)$, $A^2(4,0)$, $A^3(4,0)$, and $A^4(4,0)$. Connect the dots between these points. Predict the shape that you would get if you continued to $A^5(4,0)$, $A^6(4,0), \ldots$.

b) Plot the points $(1,0)$, $B(1,0)$, $B^2(1,0)$, $B^3(1,0)$, and $B^4(1,0)$. Connect the dots between these points. Predict the shape that you would get if you continued to $B^5(4,0)$, $B^6(4,0), \ldots$.

c) What do the eigenvalues you found in problem 1**a)** explain about your pictures from **a)** and **b)**?

d) Find complex scalars $a,\ b$ such that $(1,0) = av_1 + bv_2$, where v_1 and v_2 are the eigenvectors for B you found in problem 1**c)**.

e) Compute $B^n(1,0)$ in terms of complex exponentials.

f) Use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ to write $B^n(1,0)$ in terms of trig. functions (no complex numbers should appear in your final answer).

g) Can you predict a formula for $A^n(4,0)$ in terms of trig. functions?

5. Some quick matrix exponentials

Compute the matrix exponential e^A of:

(1) $A = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$,

(2) $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$,

(3) $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

(4) $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$,

(5) $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

6. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(t)$$

a) Write this as a matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}.$$

What is the matrix A?

b) For this matrix A, find the eigenvalues λ_1 and λ_2, as well as the eigenvectors w_1 and w_2.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, which scalars a_1 and a_2 should you choose?

d) Plug the solution with initial value $(x(0), y(0)) = (1, 1)$ to the differential equation, and confirm that it is a solution.

e) For the solution you found in **c)**, compute $(x(1), y(1))$.

7. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) Write this as a matrix differential equation $\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$.

b) Compute the eigenvalues $\lambda_1,\ \lambda_2$ and eigenvectors $v_1,\ v_2$ of the matrix A.

c) Compute the real and imaginary parts of the “eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$. This gives you two different *real* solutions to the differential equation.

d) Find the solution $(x(t), y(t))$ with initial value $(x(0), y(0)) = (1, 0)$.