Math 218D Problem Session

Week 11

1. Matrices with complex eigenvalues

a) The eigenvalues of A are $1/2i = 1/2e^{\pi/2i}$ and $-1/2i = 1/2e^{-\pi/2i}$. These eigenvalues have $|\lambda| = 1/2$ and angle $\theta = \pm\pi/2$.

The eigenvalues of B are $(1+i) = \sqrt{2}e^{\pi/4i}$ and $(1-i) = \sqrt{2}e^{-\pi/4i}$.

b) For the matrix A, an eigenvector of $(1/2)i$ is $(1,-i)$, and an eigenvector for $-(1/2)i$ is $(1,i)$.

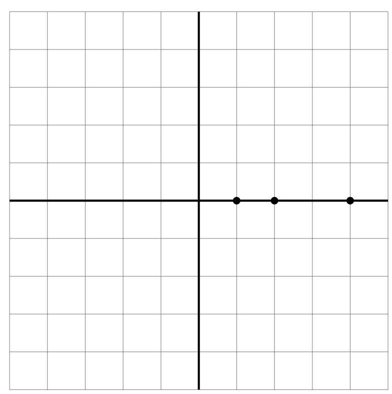
For the matrix B, an eigenvector $v_1 = (x_1, x_2)$ of B for the eigenvalue $\lambda_1 = (1+i)$ is a solution to $-ix_1 + x_2 = 0$, so we'll use $v_1 = (1,i)$ as the eigenvector.

An eigenvector for $\lambda_2 = \overline{\lambda_1}$ is $v_2 = \overline{v_1} = (1,-i)$.

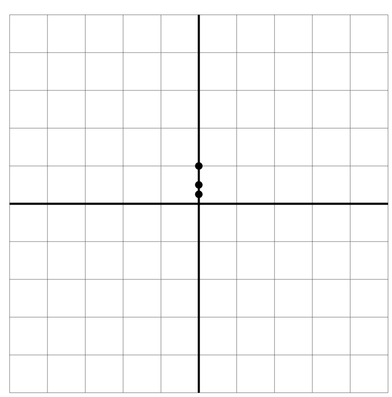
2. The dynamics of a diagonal matrix

Consider the matrix $A = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$.

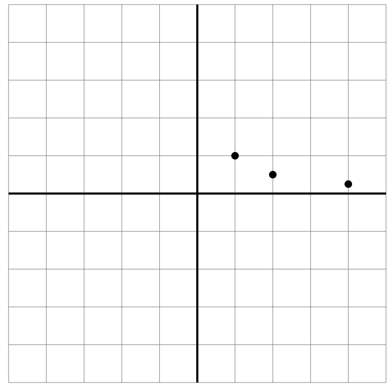
a) (1) $v = (1,0)$



(2) $v = (0,1)$



(3) $v = (1,1)$



b) We'll draw all the shapes on the same plot:

c) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches ∞, is $(1,0)$. We can see this from the picture, but can also compute this using limits. First,

$$\frac{A^n v}{||A^n v||} = \frac{2^n(1,0)+2^{-n}(0,1)}{\sqrt{2^{2n}+2^{-2n}}} = \frac{(1,0)+2^{-2n}(0,1)}{\sqrt{1+2^{-4n}}},$$

where the second equality comes from dividing both the numerator and denominator by 2^n. Since $\lim_{n\to\infty}(1,0)+2^{-2n}(0,1) = (1,0)$ and $\lim_{n\to\infty}\sqrt{1+2^{-4n}} = 1$, we find that

$$\lim_{n\to\infty}\frac{A^n v}{||A^n v||} = \frac{(1,0)}{1} = (1,0).$$

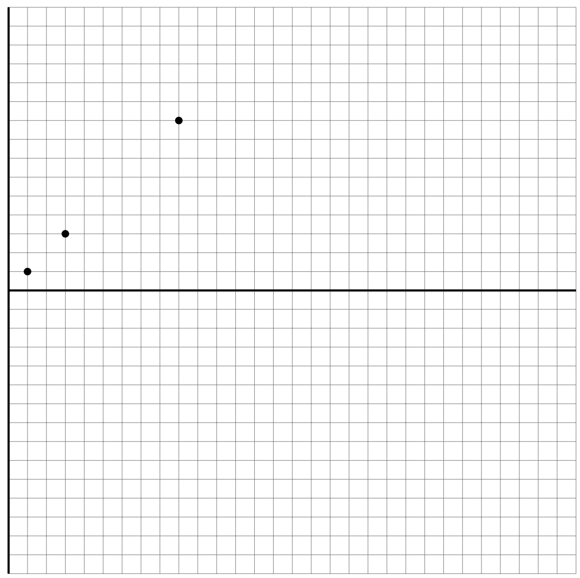
d) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches $-\infty$, is $(0,1)$.

3. The dynamics of a diagonalizable matrix

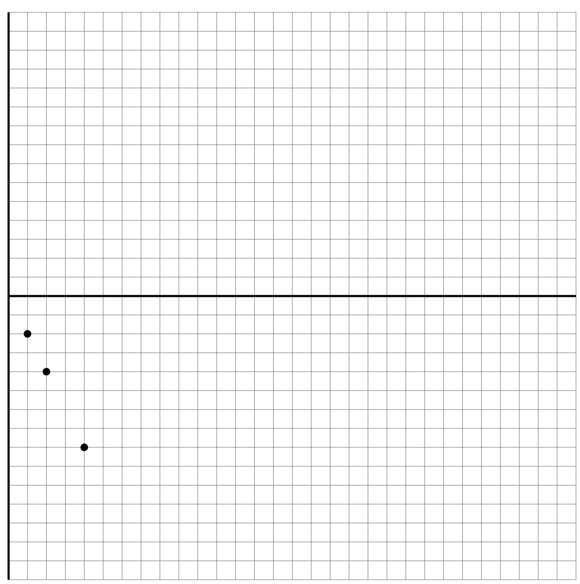
Consider the matrix A with $A(1,1) = 3(1,1)$ and $A(1,-2) = 2(1,-2)$. In other words, A is diagonalizable and you have been told the eigenvectors and eigenvalues.

a) For each of the following vectors, plot v, Av, A^2v:

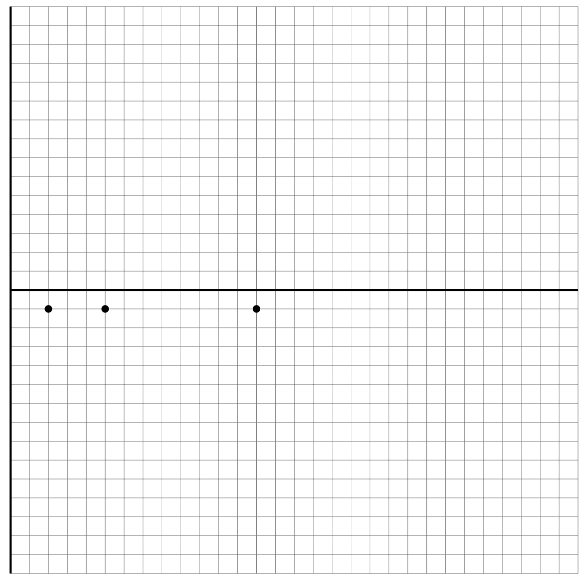
(1) $v = (1,1)$



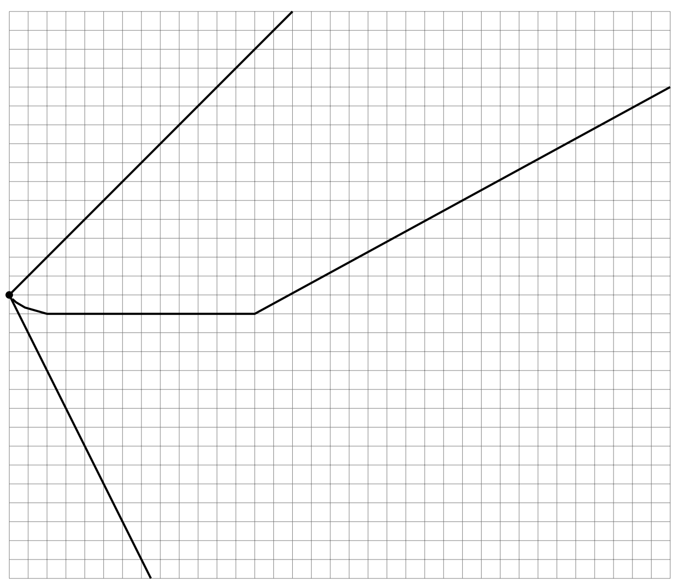
(2) $v = (1,-2)$

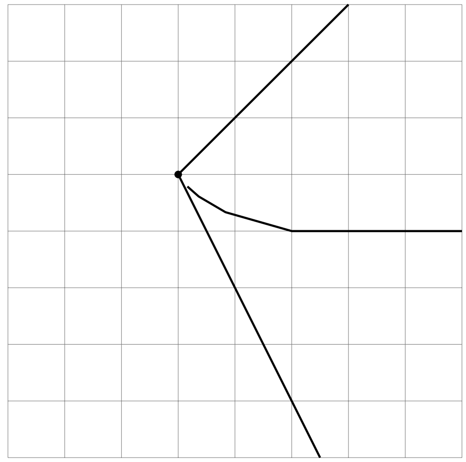


(3) $v = (2,-1)$



b) We'll draw all the shapes at once, once zoomed out and once zoomed it. The dot indicates the origin.





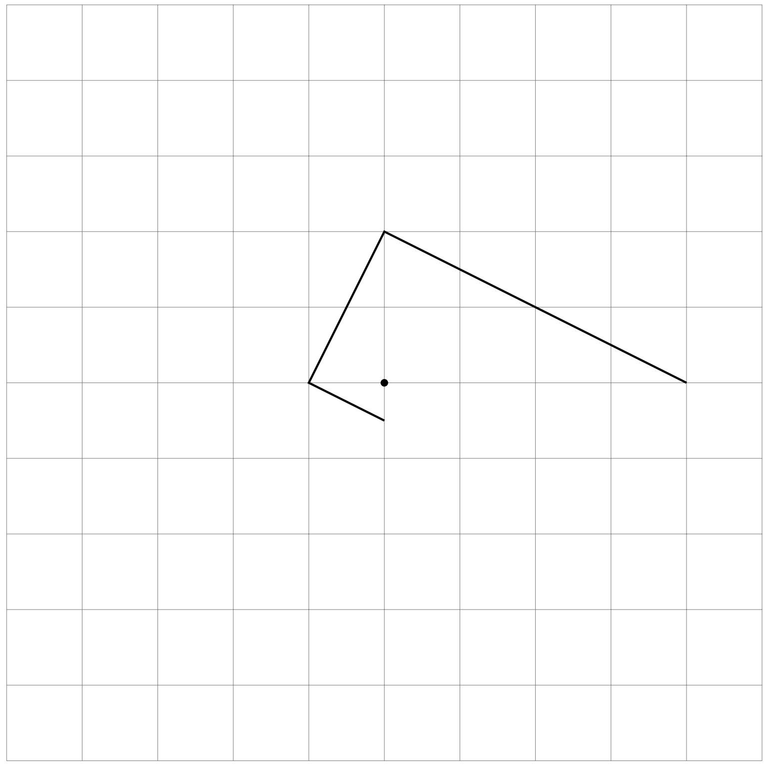
c) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches ∞, is $\frac{1}{\sqrt{2}}(1,1)$. This is not apparent from the pictures we drew, because we would need to zoom out much further to see this. But, as in Problem 1, the direction gets closer to the eigenspace of the larger eigenvalue.

d) The limit of the unit vectors $\frac{A^n v}{||A^n v||}$, as n approaches $-\infty$, is $\frac{1}{\sqrt{5}}(1,-2)$. As in Problem 1, the direction gets closer to the eigenspace of the smaller eigenvalue.

4. Dynamics with complex eigenvalues

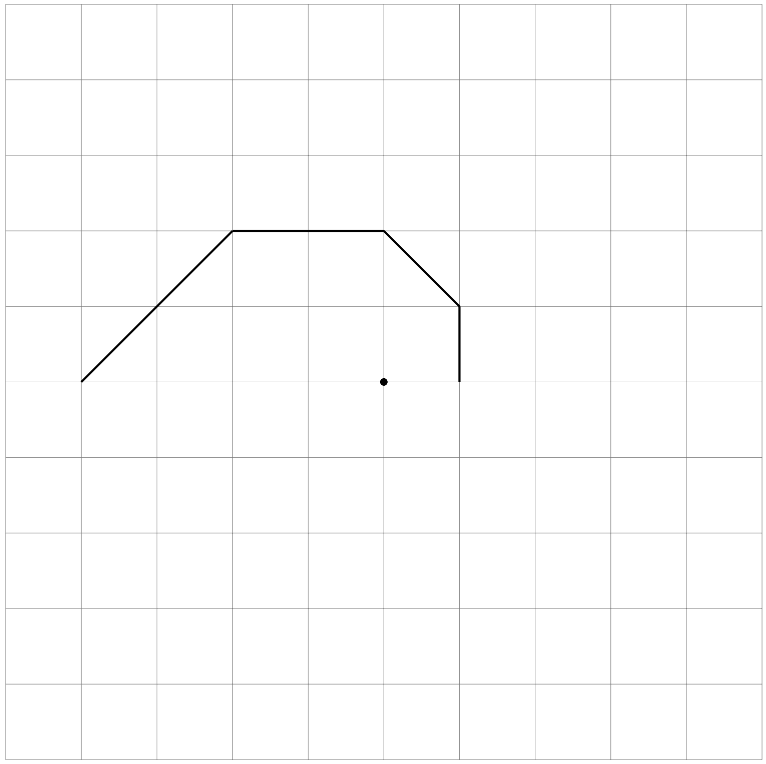
Consider the matrices $A = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

a) The points $(4,0)$, $A(4,0)$, $A^2(4,0)$, $A^3(4,0)$, and $A^4(4,0)$:



The shape is a CCW spiral in towards the origin.

b) The points $(1,0)$, $B(1,0)$, $B^2(1,0)$, $B^3(1,0)$, and $B^4(1,0)$ (and also $B^5(1,0)$, as it helps see the pattern):



c) $(1,0) = \frac{1}{2}v_1 + \frac{1}{2}v_2$ (if you had a different choice of eigenvectors, these scalars might have been complex, but with our choice they are not.)

d) $B^n(1,0) = \frac{1}{2}\lambda_1^n v_1 + \frac{1}{2}\lambda_2^n v_2$.

e) We'll use Euler's formula $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

The first component of $B^n(1,0)$ is

$$\begin{aligned}\frac{1}{2}\sqrt{2}^n(e^{n(\pi/4)i} + e^{-n(\pi/4)i}) &= \frac{1}{2}\sqrt{2}^n(\cos(n\pi/4) + i\sin(n\pi/4) + \cos(-n\pi/4) + i\sin(-n\pi/4)) \\ &= \frac{1}{2}\sqrt{2}^n(2\cos(n\pi/4) + 0i) = \sqrt{2}^n\cos(n\pi/4).\end{aligned}$$

(We used that that $\cos(-n\pi/4) = \cos(n\pi/4)$ while $\sin(n\pi/4) = -\sin(n\pi/4)$.)

The second component of $B^n(1,0)$ is

$$\begin{aligned}\frac{1}{2}\sqrt{2}^n(ie^{n(\pi/4)i} - ie^{-n(\pi/4)i}) &= \frac{1}{2}\sqrt{2}^n(i\cos(n\pi/4) - \sin(n\pi/4) - i\cos(-n\pi/4) + \sin(-n\pi/4)) \\ &= \sqrt{2}^n\sin(n\pi/4).\end{aligned}$$

Therefore the vector $B^n(1,0)$ has closed form

$$B^n(1,0) = (\sqrt{2}^n\cos(n\pi/4), \sqrt{2}^n\sin(n\pi/4)).$$

f) Based on the eigenvalues for A and the picture, we might guess

$$A^n(4,0) = (4(1/2)^n\cos(n\pi/2), 4(1/2)^n\sin(n\pi/2)).$$

5. Some quick matrix exponentials

(1) $A=\begin{pmatrix}2&0\\0&-3\end{pmatrix}$, $e^A=\begin{pmatrix}e^2&0\\0&e^{-3}\end{pmatrix}$.

(2) $A=\begin{pmatrix}1&1\\0&-1\end{pmatrix}$. A has diagonalization $A=\begin{pmatrix}1&-1\\0&2\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1&-1\\0&2\end{pmatrix}^{-1}$, so $e^A=e^{CDC^{-1}}=Ce^DC^{-1}=C\begin{pmatrix}1&0\\0&-1\end{pmatrix}C^{-1}$, which you can multiply to get the final answer.

(3) $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $e^A=I+A+A^2/2+\cdots=I+A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

(4) $A=\begin{pmatrix}1&2\\0&1\end{pmatrix}$, $e^A=e^{\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&2\\0&0\end{pmatrix}}=e^{\begin{pmatrix}1&0\\0&1\end{pmatrix}}\cdot e^{\begin{pmatrix}0&2\\0&0\end{pmatrix}}=\begin{pmatrix}e&0\\0&e\end{pmatrix}\begin{pmatrix}1&2\\0&1\end{pmatrix}=\begin{pmatrix}e&2e\\0&e\end{pmatrix}$.

(5) $A=\begin{pmatrix}3&1&0\\0&3&1\\0&0&3\end{pmatrix}$, $e^A=e^{3I}\cdot e^{\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}}=e^3(I+\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}+\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}^2/2)=\begin{pmatrix}e^3&e^3&e^3/2\\0&e^3&e^3\\0&0&e^3\end{pmatrix}$.

6. A differential equation

Consider the system of differential equations

$$x'(t) = 3x(t) + 2y(t)$$
$$y'(t) = 4x(t) - 4y(y)$$

a) The matrix A in the matrix differential equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

is $A = \begin{pmatrix} 3 & 2 \\ 4 & -4 \end{pmatrix}$.

b) This matrix A has characteristic polynomial $\lambda^2 + \lambda - 20$, with eigenvalues $\lambda_1 = -5$ and $\lambda_2 = 4$. The eigenvectors are $w_1 = (-2, 8)$ and $w_2 = (-1, 2)$.

c) Every solution is of the form $(x(t), y(t)) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. If you want the solution to have initial value $(x(0), y(0)) = (1, 1)$, your scalars must solve $(1, 1) = a_1 w_1 + a_2 w_2$. You can solve this by solving the system of linear equations $\left(\begin{array}{cc|c} -2 & -1 & 1 \\ 8 & 2 & 1 \end{array}\right)$. This has solution $a_1 = -5/4, a_2 = 3/2$, i.e. $-5/4(-2, 8) + 3/2(-1, 2) = (1, 1)$.

d) The solution is $u(t) = a_1 e^{\lambda_1 t} w_1 + a_2 e^{\lambda_2 t} w_2$. When we plug this into the differential equation, we get $u'(t) = a_1 \lambda_1 e^{\lambda_1 t} w_1 + a_2 \lambda_2 e^{\lambda_2 t} w_2$ on one side, and $Au(t) = a_1 e^{\lambda_1 t}(\lambda_1 w_1) + a_2 \lambda_2 e^{\lambda_2 t}(\lambda_2 w_2)$ on the other. Since these are equal, $u(t)$ solves the differential equation. (We didn't actually need to use the values for a_1 and a_2 to check this.)

e) The value of $(x(1), y(1))$ is $-5/4 e^{\lambda_1} w_1 + 3/2 e^{\lambda_2} w_2 = (5/2 e^{-5} - 3/2 e^4, -10 e^{-5} + 3e^4)$. You don't need to simplify any further than this.

7. A complex ODE

Consider the system of differential equations

$$x'(t) = x(t) - y(t),$$
$$y'(t) = x(t) + y(t).$$

a) The matrix differential equation would be

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

with $A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

b) The characteristic polynomial is $(\lambda - 1)^2 + 1 = 0$, which gives the eigenvalues $\lambda_1 = 1 + i$ and $\lambda_1 = 1 - i$. The associated eigenvectors are $w_1 = (i, 1)$ and $w_2 = (-i, 1)$.

c) The eigenvector solution" $(x(t), y(t)) = e^{\lambda_1 t} v_1$ is

$$e^{t(1+i)} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^t \left(\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \right),$$

from which we can observe the real and imaginary part.

d) We write

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\frac{i}{2} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Hence the solution would be

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = -\frac{i}{2} e^{(1+i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} + \frac{i}{2} e^{(1-i)t} \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$