Math 218D Problem Session

Week 12

1. Shape of quadratic forms

For each of the following quadratic forms:

(1) Plot the equation $q(x, y) = 1$ using a computer, and describe the shape (for example, for **a)** you should get an ellipse in $\mathbf{R}^2$, not an elliptic paraboloid in $\mathbf{R}^3$).

(2) Find the 2×2 symmetric matrix $S = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ such that

$$q(x,y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix} = ax^2 + 2bxy + cy^2.$$

(3) Recall that a symmetric matrix is **positive-definite** if all of its eigenvalues are positive. Test if the symmetric matrix S is positive-definite or not using the **pivot test**: Put S into REF without doing row-swaps or scaling. (If you need to do a row-swap, the matrix is not positive-definite.) If the diagonal entries of the REF are all positive, then S is positive-definite.

(4) What does the positive-definiteness of S have to do with the shape from (1)? You may need to do many examples until you see the pattern.

a) $q(x, y) = 2x^2 + 3y^2$

b) $q(x, y) = x^2 - 5y^2$

c) $q(x, y) = y^2$

d) $q(x, y) = -3x^2 - 2y^2$

e) $q(x, y) = x^2 + 3xy + y^2$

f) $q(x, y) = 2x^2 + 4xy + y^2$

g) $q(x, y) = x^2 - 4xy + 5y^2$

Our final two quadratic forms are in 3 variables: this means that S is a 3×3 matrix, and $q(x, y, z) = \begin{pmatrix} x & y & z \end{pmatrix} S \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

h) $q(x, y, z) = x^2 + y^2 + z^2 + xy + yz + xz$

i) $q(x, y, z) = x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$

2. Diagonalizing quadratic forms

Consider the quadratic form

$$q(x,y) = \frac{5}{2}x^2 + 3xy + \frac{5}{2}y^2.$$

a) What is the symmetric matrix S so that $q(x,y) = \begin{pmatrix} x & y \end{pmatrix} S \begin{pmatrix} x \\ y \end{pmatrix}$?

b) Find the symmetric diagonalization, $S = QDQ^T$, where the matrix Q is orthonormal. The columns of Q are orthonormal eigenvectors v_1 and v_2, with eigenvalues λ_1 and λ_2.

c) The quadratic form for the diagonal matrix D is $\begin{pmatrix} x_0 & y_0 \end{pmatrix} D \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda_1^2 x_0^2 + \lambda_2^2 y_0^2$. Plot $q(x,y) = 1$ and $\lambda_1 x_0^2 + \lambda_2 y_0^2 = 1$. What is the geometric relationship between these shapes?

d) Confirm that $q(x,y) = \lambda_1 x_0^2 + \lambda_2 y_0^2$, where we relate the variables x_0 and y_0 to the variables x and y using $\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = Q^T \begin{pmatrix} x \\ y \end{pmatrix}$.

e) Using **d)**, explain why the equation $q(x,y) = 1$ describes an ellipse. How does this relate to the **eigenvalue test** for positive-definite matrices?

f) Using **d)**, explain why the function $q(x,y)$ is always positive (unless $x = y = 0$). How does this relate to the **positive-energy test** for positive-definite matrices?

g) What are the lengths of the major and minor axes of the ellipse $q(x,y) = 1$?
Hint: What are the lengths of the major and minor axes of the ellipse $\lambda_1 x_0^2 + \lambda_2 y_0^2 = 1$?

h) What are the directions of the major and minor axes of the ellipse $q(x,y) = 1$?

i) Consider the **constrained optimization** problem: what is the maximum value of the function $q(x,y)$ on the unit circle $x^2 + y^2 = 1$? at what point (x,y) on the unit circle does q achieve the maximum?
Hint: First answer these questions for the quadratic form $\lambda_1 x_0^2 + \lambda_2 y_0^2$.

3. LDL^T **decomposition**

Find the LDL^T decomposition of the following positive-definite symmetric matrices, by:

(1) Computing the $A = LU$ decomposition.

(2) Setting $D =$ the diagonal matrix with same diagonal entries as U.

a) $S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

b) $S = \begin{pmatrix} 4 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 4 \end{pmatrix}$

4. Relation to the quadratic formula

For 2×2 symmetric matrices $S = \begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}$, there is an easy test for positive-definiteness, the **discriminant test**:

$$S \text{ is positive-definite if and only if both } a > 0 \text{ and } b^2 - 4ac < 0.$$

Let's verify this test in two ways, by relating it to other tests.

a) Method one: Relate the discriminant test to the **determinant test**: S is positive-definite if and only if $\det((a)) > 0$ and $\det(\begin{pmatrix} a & \frac{1}{2}b \\ \frac{1}{2}b & c \end{pmatrix}) > 0$.

b) Method two:

(1) Show that the quadratic form $q(x,y) = (x,y)^T S(x,y)$ equals

$$q(x,y) = ax^2 + bxy + cy^2$$

and factors into

$$q(x,y) = a(x - \frac{-b + \sqrt{b^2 - 4ac}}{2} y)(x - \frac{-b - \sqrt{b^2 - 4ac}}{2} y).$$

(2) If $b^2 - 4ac < 0$, explain why $q(x,y) \neq 0$ for all real numbers x, y (not both zero).

This means that either $q(x,y) > 0$ for all $(x,y) \neq (0,0)$ or $q(x,y) < 0$ for all $(x,y) \neq (0,0)$.

(3) If both $a > 0$ and $b^2 - 4ac < 0$, explain why $q(x,y) > 0$ for all real numbers x, y (not both zero).
Hint: If $a > 0$, can you find a point (x,y) where $q(x,y) > 0$?

This show that **if S satisfies the discriminant test, it satisfies the energy test.**