# Math 218D Problem Session

Week 2

## 1. Row Echelon Form

**a)**

$$\left(\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 3 & 2 \end{array}\right).$$

In REF, pivots are 1 and 3

**b)**

$$\begin{aligned} 5x - \phantom{0}y &= 1 \\ 0x + 0y &= 6 \\ 0x + 0y &= 0 \end{aligned}.$$

In REF, pivots are 5 and 6

**c)**

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 3 & 1 & 2 \\ 2 & 0 & 0 & 1 \end{array}\right).$$

Not in REF

**d)**

$$\begin{aligned} 2x + y &= 3 \\ y &= 5 \end{aligned}.$$

In REF, pivots are 2 and 1

**e)**

$$\begin{aligned} 3x + 2y + z &= 0 \\ 0x + 0y + 0z &= 0 \\ -x - 2y + 4z &= 1 \end{aligned}.$$

Not in REF

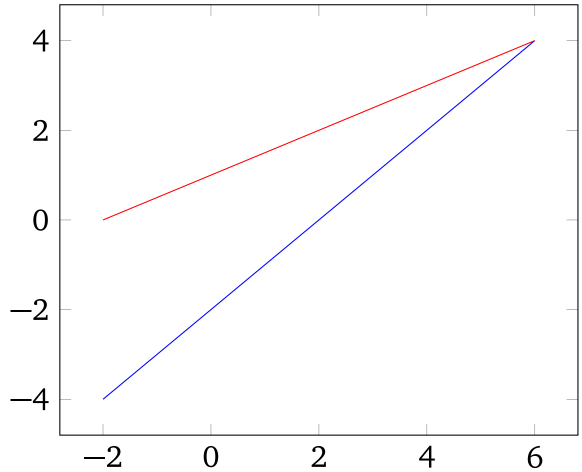
**f)**

$$\begin{aligned} 5x + 5y &= 5 \\ x + y &= 1 \end{aligned}.$$

Not in REF

## 2. Two Equations and Two Unknowns

**a)**




**b)** The linear system is

$$\begin{aligned} x - \phantom{4}y &= \phantom{-}2 \\ 2x - 4y &= -4. \end{aligned}$$

Subtract $2 \cdot R_1$ from $R_2$ to obtain:

$$\begin{aligned} x - y &= 2 \\ -2y &= -8. \end{aligned}$$

**c)** Divide the second row by 2 to obtain:

$$\begin{aligned} x - y &= 2 \\ y &= 4. \end{aligned}$$

**d)** Add the second row to the first row to obtain:

$$\begin{aligned} x &= 6 \\ y &= 4. \end{aligned}$$

This is the solution.

**e)** $6-4=2,\ 2\cdot 6-4\cdot 4=-5.$

**f)** The system first becomes in REF after the 1st row operation. The pivots are 1 and $-2$.

## 3. Three Equations Three Unknowns

**a)** $A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}$.

**b)** The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 2 & -8 & 8 & -2 \\ -6 & 3 & -15 & 9 \end{array}\right).$$

**c)** First, replace $R_2$ by $R_2 - 2R_1$ ($R_2 \mathrel{+}= -2R_1$).

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ -6 & 3 & -15 & 9 \end{array}\right).$$

Then $R_3 \mathrel{+}= 6R_1$:

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & -2 & 6 & -10 \\ 0 & -15 & -9 & 33 \end{array}\right)..$$

Now, you can do row scaling here, although you don't need to. Let's do it now to simplify our rows: $R_2 \times= -(1/2)$ and $R_3 \times= -(1/3)$ (combining two elementary operations at once):

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 5 & 3 & -11 \end{array}\right).$$

We do one more row addition, replacing $R_2$ with $R_2 - 5R_1$ ($R_2 \mathrel{-}= 5R_1$):

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 18 & -36 \end{array}\right).$$

Do one more row scaling, replacing $R_3$ with $\frac{1}{18}R_3$ ($R_3 \times= 1/18$):

$$\left(\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -2 \end{array}\right).$$

**d)** I used 6 elementary row operations, but the row scalings could have been avoided, giving you as few as 3.

**e)** The system of equations is now

$$\begin{aligned} x_1 - 3x_2 + x_3 &= 4 \\ x_2 - \quad 3x_3 &= 5 \\ x_3 &= -2 \end{aligned} \quad .$$

Substituting $x_3 = -2$, we obtain the system

$$\begin{aligned} x_1 - 3x_2 &= 6 \\ x_2 &= -1 \\ x_3 &= -2 \end{aligned} \quad .$$

Substituting $x_2 = -1$, we obtain the system

$$\begin{aligned} x_1 &= 3 \\ x_2 &= -1 \\ x_3 &= -2 \end{aligned} ,$$

which is the solution.

**f)** Check $\begin{pmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{pmatrix} \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -2 \\ 9 \end{pmatrix}$.

## 4. Another One—What's Different?

Consider the system of three linear equations

$$\begin{aligned} x_1 - 2x_2 + \phantom{8}x_3 &= -2 \\ 2x_1 - 4x_2 + 8x_3 &= \phantom{-}2 \\ x_1 - 3x_2 - \phantom{8}x_3 &= \phantom{-}1. \end{aligned}$$

**a)** The linear system is

$$\begin{aligned} x_1 - 2x_2 + \phantom{8}x_3 &= -2 \\ 2x_1 - 4x_2 + 8x_3 &= \phantom{-}2 \\ x_1 - 3x_2 - \phantom{8}x_3 &= \phantom{-}1. \end{aligned}$$

By doing two row subtraction operations ($R_2 - = 2R_1$ and $R_3 - = R_1$), we obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ 6x_3 &= 6 \\ -x_2 - 2x_3 &= 3 \end{aligned} \quad .$$

**b)** We swap rows 1 and 2 to obtain

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -2 \\ -x_2 - 2x_3 &= 3 \\ 6x_3 &= 6 \end{aligned} \quad .$$

**c)** Dividing row 3 by 6 gives $x_3 = 1$, which we substitute into the first two equations:

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ -x_2 &= 5 \\ x_3 &= 1. \end{aligned}$$

Dividing row 2 by $-1$ gives $x_2 = -5$, which we substitute into the 1st equation:

$$\begin{aligned} x_1 &= -13 \\ x_2 &= -5 \\ x_3 &= 1. \end{aligned}$$

This is the solution.

**5. Traffic Jam**

**a)** We start with

$$\begin{aligned} 120 + w &= 250 + x \\ 120 + x &= \phantom{0}70 + y \\ 390 + y &= 250 + z \\ 115 + z &= 175 + w \end{aligned}$$

or

$$\begin{aligned} x - w &= -130 \\ -x + y &= 50 \\ -y + z &= 140 \\ -z + w &= -60. \end{aligned}$$

**b)** Eliminating $x$ from the second equation gives

$$\begin{aligned} x - w &= -130 \\ y - w &= -80 \\ -y + z &= 140 \\ -z + w &= -60. \end{aligned}$$

**c)** Eliminating $y$ from the third equation gives

$$\begin{aligned} x - w &= -130 \\ y - w &= -80 \\ z - w &= 60 \\ -z + w &= -60. \end{aligned}$$

**d)** Eliminating $z$ from the fourth equation gives

$$\begin{aligned} x - w &= -130 \\ y - w &= -80 \\ z - w &= 60 \\ 0 + 0 &= 0. \end{aligned}$$

**e)** We can't just use substitution, as our final equation is not of the form $w = (?)$. The number of cars on roads $x, y$, and $z$ all depend on how many cars are on $w$.

**f)** The system has infinitely many solutions. There can be as many cars as you want, travelling in a circle around the town.

**g)** The augmented matrix is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -130 \\ 0 & 1 & 0 & -1 & -80 \\ 0 & 0 & 1 & -1 & 60 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The pivots are the 1's. Not every row has a pivot. The fourth column does not have a pivot - as we will discuss in week 3, this means that we can find a solution which makes the fourth variable take any value we want. Such a variable is called a *free variable*.

## 6. Reduced Row Echelon Form

(1) No; Yes; Yes; Yes.

(2) No; No; No; No.

(3) (a) Use Gaussian elimination to obtain $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

(b) Use Jordan substitution to obtain $\left(\begin{array}{ccc|r} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & 1 & 2 \end{array}\right)$.

(c) Use Jordan substitution to obtain $\left(\begin{array}{cc|r} 1 & 0 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$.

(d) Use Jordan substitution to obtain $\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{array}\right)$.

(4) (a) Pivots: $1, 1, 1$. No solution.

(b) Pivots: $1, 2, 1$. Unique solution.

(c) Pivots: $2, -1$. Unique solution.

(d) Pivots: $1, 1$. No solution.