Math 218D Problem Session

Week 3

1. Elementary matrices

a) $\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

c) The matrix is $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

2. True or false?

a) Yes, since we can “un-do" the row operation.

b) Yes, since the number of pivots can not be larger than the number of rows.

c) No, consider $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

3. Solving $Ax = b$ using $A = LU$

When you know the LU factorization of a matrix A, you can use it to solve the matrix equation $Ax = b$. In this problem we will go through this process in an example.
Solve the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix},$$

using the $A = LU$ decomposition

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

b)

$$\begin{aligned} c_1 \qquad\qquad &= 1 \\ c_1 + \ c_2 \qquad &= 2 \\ 2c_1 + 0c_2 + c_3 &= 3 \end{aligned},$$

and substitution gives $(c_1, c_2, c_3) = (1, 1, 1)$.

c)

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ x_3 &= 1 \end{aligned},$$

and substitution gives $(x_1, x_2, x_3) = (1, -1, 1)$.

d) Check $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

4. Finding $A = LU$ and A^{-1} using elementary matrices

a) $A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 4 \\ 1 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 4 & 6 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

The three row operations, in order, are $R_2 -= 2R_1$, $R_3 -= R_1$, $R_3 -= 5R_2$.

We have computed $U = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

b) The elementary matrices are

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, \ E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}.$$

Therefore

$$U = E_3 E_2 E_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A.$$

c)

$$A = E_1^{-1} E_2^{-1} E_3^{-1} U = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} U.$$

d)

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 5 & 1 \end{pmatrix}.$$

e)

$$U = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The three row operations are $R_3 \times 1/4$, $R_1 -= 2R_3$, $R_1 += R_2$, corresponding to the elementary matrices

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix},\ E_5 = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ E_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using these matrices,

$$E_6E_5E_4U = I_3.$$

f)

$$A^{-1} = E_6E_5E_4E_3E_2E_1.$$

g)

$$\left(A \mid I_3\right) = \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 4 & 0 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 1 & 4 & 6 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 5 & 4 & -1 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 9 & -5 & 1 \end{array}\right) = \left(U \mid E_3E_2E_1\right).$$

We are halfway done - the right half of this matrix is now $E_3E_2E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 9 & -5 & 1 \end{pmatrix}$.

An aside: one convenient fact about $L = E_1^{-1}E_2^{-1}E_3^{-1}$ is the way in which its entries precisely correspond to the row operations performed. It is harder to interpret the entries of $E_3E_2E_1$. For example, why does 9 appear in $E_3E_2E_1$? It is because

$$\text{final } R_3 = 9(\text{original } R_1) - 5(\text{original } R_2) + (\text{original } R_3).$$

Continuing onwards:

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 9 & -5 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & -7/2 & -5/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11/2 & -3/2 & -1/2 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 9/4 & -5/4 & 1/4 \end{array}\right) = \left(I_3 \mid A^{-1}\right).$$

We conclude that $A^{-1} = \begin{pmatrix} -11/2 & -3/2 & -1/2 \\ -2 & 1 & 0 \\ 9/4 & -5/4 & 1/4 \end{pmatrix}$.

problem**Finding** $PA = LU$

h) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$.

i) $L. = \begin{pmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -20 \\ 0 & 0 & -15 \end{pmatrix}$.

j) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.1 & -0.2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{pmatrix}$.

5. Finding $PA = LU$

a) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$.

b) $L. = \begin{pmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 5 & -1 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -10 & -20 \\ 0 & 0 & -15 \end{pmatrix}$.

c) $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ -0.1 & -0.2 & 1 \end{pmatrix}$, $U = \begin{pmatrix} -10 & -20 & -30 \\ 0 & 5 & -5 \\ 0 & 0 & -3 \end{pmatrix}$.

6. Solving $Ax = b$ using $PA = LU$

a) $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$, $U = \begin{pmatrix} 2 & 2 & 5 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{3}{2} \end{pmatrix}$.

b) $Pb = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$. P swaps the first and second raws, and then swap the second and third rows of b.

c)

$$\begin{array}{rcl} c_1 & & = 5 \\ \frac{1}{2}c_1 + c_2 & & = 2 \\ \frac{1}{2}c_1 + 0c_2 + c_3 & = & 1 \end{array},$$

and substitution gives $(c_1, c_2, c_3) = (5, -\frac{1}{2}, -\frac{3}{2})$.

d)

$$\begin{array}{rcl} 2x_1 + 2x_2 + 5x_3 & = & 5 \\ x_2 + \frac{1}{2}x_3 & = & -\frac{1}{2} \\ -\frac{3}{2}x_3 & = & -\frac{3}{2} \end{array},$$

and substitution gives $(x_1, x_2, x_3) = (1, -1, 1)$.

e) Check $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$.