Math 218D Problem Session

Week 4 Solutions

1. Parametric forms

a) The RREF is $\left(\begin{array}{ccc|c} 1 & 0 & 1/3 & 1/3 \\ 0 & 1 & 2/3 & -1/3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) The parametric form of the solution is:

$$\begin{aligned} x_1 &= -\tfrac{1}{3}x_3 + \tfrac{1}{3} \\ x_2 &= -\tfrac{2}{3}x_3 - \tfrac{1}{3} \\ x_3 &= \ x_3 \end{aligned} .$$

Setting $x_3 = 0$ gives one solution: $(x_1, x_2, x_3) = (1/3, -1/3, 0)$. Setting $x_3 = 1$ gives another solution $(x_1, x_2, x_3 = (0, -1, 1)$.

c) The parametric vector form is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/3 \\ -1/3 \\ 0 \end{pmatrix}.$$

d) The line passes through the point $(1/3, -1/3, 0)$, and goes in the direction of the vector $(-1/3, -2/3, 1)$.

e) This system of equations has no solutions.

f) The parametric vector form of this system is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -1/3 \\ -2/3 \\ 1 \end{pmatrix}.$$

The solution to the homogeneous equation is a line, parallel to the line from part d), passing through the origin.

g) A vector $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ makes $Ax = b$ consistent precisely when $-b_1 + b_2 - b_3 = 0$.

h) The span of these vectors is the same as the set of vectors making $Ax = b$ consistent. By g), this is the same as the vectors which satisfying a single linear equation. The set of vectors satisfying a single linear equation is a plane.

2. Parallel lines

a) $\text{Span}\{(-2,1,1)\}+(3,-2,0)$

b) One possible answer is $P_1=(3,-2,0)$, $P_2=(1,-1,1)$, $P_2-P_1=(-2,1,1)$.

c) To get a parallel line, you need the same matrix A but a different b vector. You can find the correct b vector by multiplying A times $x=(1,1,1)$: $A(1,1,1)=(3,6)$. In other words, the solution set of

$$\begin{aligned} x + y + z &= 3 \\ 2x + 3y + z &= 6 \end{aligned}$$

is parallel to L and passing through $(1,1,1)$.

3. The geometry of spans

a) No, it is not possible. You can confirm this by computing the RREF of $\left(\begin{array}{cc|c} 1 & 1 & 1 \\ -1 & -1 & 1 \\ 5 & 1 & 0 \end{array}\right)$.

Alternately, you could observe that the first two components of $\begin{pmatrix}1\\-1\\5\end{pmatrix}$ and $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ add up to 0, while the first two components of $\begin{pmatrix}1\\1\\0\end{pmatrix}$ do not.

b) It is all of $\mathbf{R}^3$, since $\begin{pmatrix}1\\1\\0\end{pmatrix}$ is not contained in the plane $\text{Span}\left\{\begin{pmatrix}1\\-1\\5\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}\right\}$ (using 3a)).

c) By computing the REF of $\left(\begin{array}{cc|c} 1 & 1 & b_1 \\ -1 & -1 & b_2 \\ 5 & 1 & b_3 \end{array}\right)$, we confirm that the vectors $b=(b_1,b_2,b_3)$ which make

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \\ 5 & 1 \end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix}$$

consistent are precisely those where $b_1+b_2=0$. This means that the plane parametrized by

$$x_1\begin{pmatrix}1\\-1\\5\end{pmatrix}+x_2\begin{pmatrix}1\\-1\\1\end{pmatrix}$$

has equation

$$x+y=0.$$

d) Yes, you can find scalars x_1, x_2 so that

$$x_1 \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},$$

since $(4, -4, 0)$ solves the equation $x + y = 0$ found in 3c).

e) The vectors $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ are not parallel, so they span a plane. The third vector $\begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix}$ is contained in that plane by 3d), so adding it to the list of vectors does not enlarge the span.