Math 218D Problem Session

Week 6

1. Subspaces?

Decide if each of the following sets of vectors is or is not a subspace, and explain why or why not.

a)

$$\{(x, y, z) \in \mathbf{R}^3 \mid x + y = 1 - z\}$$

b)

$$\{(x, y) \in \mathbf{R}^2 \mid x - 2y = 0\}$$

c) For A a 3×3 matrix, the set

$$\left\{ v \in \mathbf{R}^3 \mid Av = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

d) The set

$$\left\{ (x, y) \in \mathbf{R}^2 \mid \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \right\}$$

e)

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

f)

$$\{(x, y) \in \mathbf{R}^2 \mid x^2 + 2xy + y^2 = 0\}$$

The 4 *fundamental subspaces* associated to a matrix A are Nul(A), Col(A), Nul(A^T), and Col(A^T).

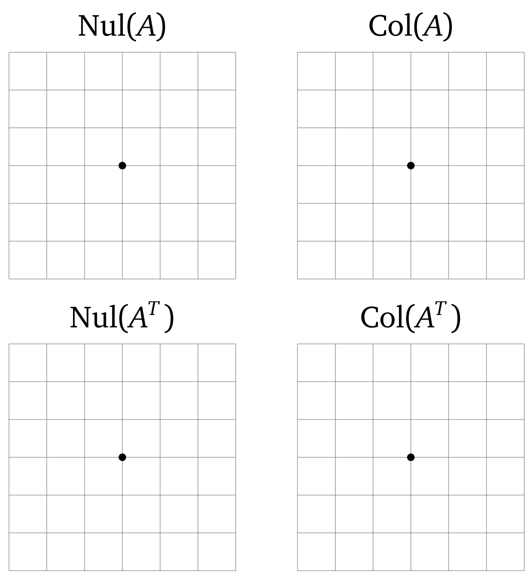
A spanning set for the null space Nul(A) can be found by finding the parametrized vector form of the solution set of $Ax = 0$.

2. The fundamental subspaces I

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

a) Find a spanning set for each of the four fundamental subspaces of this matrix A.

b) Draw each of the fundamental subspaces:



c) Compute dim(Nul(A)) + dim(Col(A)), where dim refers to the *dimension* of the subspace. The dimension of a point, line, or plane is 0, 1, or 2.

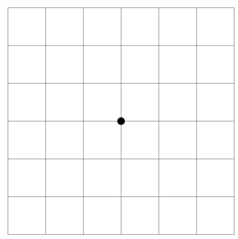
3. The fundamental subspaces II

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

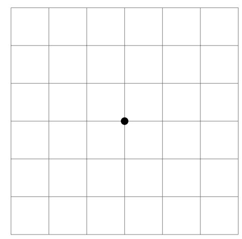
a) Find a spanning set for each of the four fundamental subspaces of the matrix A.

b) Draw each of the fundamental subspaces:

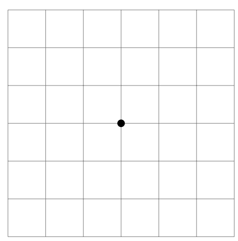
Nul(A)



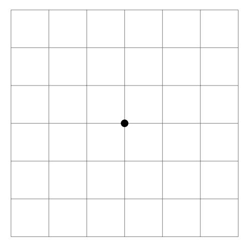
Col(A)



Nul(A^T)



Col(A^T)



c) Compute $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T))$.

d) Describe a geometric relationship between Nul(A) and Col(A^T). Then describe the relationship between Col(A) and Nul(A^T).

4. The fundamental subspaces III

Consider the matrix

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -2 & 2 & -4 \end{pmatrix}.$$

a) Is $\mathrm{Col}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

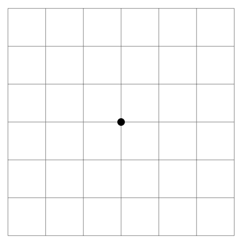
b) Is $\mathrm{Nul}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

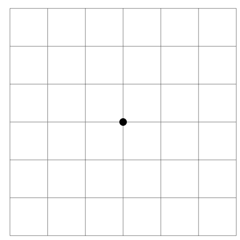
c) Is $\mathrm{Col}(A)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

d) Is $\mathrm{Nul}(A^T)$ a subspace of $\mathbf{R}^2$ or $\mathbf{R}^3$?

e) Two of the four subspaces are contained in $\mathbf{R}^2$. Draw these two subspaces, and describe the geometric relationship between them.

____? ____?





f) Two of the four subspaces are contained in $\mathbf{R}^3$. For this matrix, one is a line and the other is a plane. Determine which is which.

g) Find a vector whose span is the line.

h) Find two vectors whose span is the plane.

i) Find an equation $a_1x + a_2y + a_3z = 0$ for the plane.

Hint: Make the two vectors from **h)** into the columns of a matrix B. Find an equation which b_1, b_2, b_3 must satisfy in order for $B\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ to be consistent. Why does this answer the question?

j) What can you observe about the relationship between the answers to **g)** and **i)**? What does this mean geometrically?

5. Linear (in)dependence

a) Are the vectors $\binom{1}{0}$, $\binom{1}{-1}$ linearly independent? If not, write down a linear dependence relation.

b) Are the vectors $\binom{1}{1}$, $\binom{1}{-1}$, $\binom{3}{2}$ linearly independent? If not, write down a linear dependence relation.

c) What is the dimension of Span $\left\{ \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-2\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\}$? Why?

d) Consider 2 linearly independent vectors $u, v \in \mathbf{R}^n$. Show that the two vectors $u+v$, $u-v$ are linearly independent.

e) Consider 3 vectors $u, v, w \in \mathbf{R}^n$. Show that the three vectors $u+v$, $u+2v-w$, $v-w$ are linearly *dependent*.

f) Show that the vectors

$$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\\-4\end{pmatrix}, \begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

are linearly dependent, by writing down a linear dependence relation among them.

Hint: Write down the matrix A whose columns are these vectors, and find a non-zero vector in Nul(A). Why does this solve the question?

6. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) Which of the subspaces Row(A), Col(A), Nul(A), Nul(A^T) can you find a basis for? If you *can* find a basis, do.

b) Which of the subspaces Row(A), Col(A), Nul(A), Nul(A^T) can you find the dimension of? If you *can* find a dimension, do.