Math 218D Problem Session

Week 6

1. Subspaces?

a) Not a subspace, since it doesn't contain $(0,0,0)$.

b) A subspace, since it is the solution set of a homogeneous linear equation.

c) Not a subspace, since it doesn't contain $(0,0,0)$.

d) A subspace, since it is the left-null space of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \end{pmatrix}$

e) Not a subspace, since it doesn't contain $(0,0,0)$.

f) A subspace, since

$$\left\{(x,y)\in \mathbf{R}^2 : x^2+2xy+y^2=0\right\} = \left\{(x,y)\in \mathbf{R}^2 : (x+y)^2=0\right\} = \left\{(x,y)\in \mathbf{R}^2 : x+y=0\right\}.$$

2. The fundamental subspaces I

a) The $\mathrm{Nul}(A)$ and $\mathrm{Nul}(A^T)$ are points, while the $\mathrm{Col}(A)$ and $\mathrm{Col}(A^T)$ are all of $\mathbf{R}^2$.

b) $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T)) = 2$

3. The fundamental subspaces II

a) The spanning sets are $\mathrm{Col}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$, $\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$, $\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$, $\mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, although other answers are possible.

b) Draw the lines spanned by the vectors of **a)**.

c) $\dim(\mathrm{Nul}(A)) + \dim(\mathrm{Col}(A^T)) = 2$.

d) The lines $\mathrm{Nul}(A)$ and $\mathrm{Col}(A^T)$ are perpendicular. The lines $\mathrm{Col}(A)$ and $\mathrm{Nul}(A^T)$ are perpendicular.

4. The fundamental subspaces III

a) $\mathrm{Col}(A^T)$ is a subspace of $\mathbf{R}^3$

b) $\mathrm{Nul}(A)$ is a subspace of $\mathbf{R}^3$

c) $\mathrm{Col}(A)$ is a subspace of $\mathbf{R}^2$

d) $\mathrm{Nul}(A^T)$ is a subspace of $\mathbf{R}^2$

e) The column space is the line spanned by $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, and the left-null space is the line spanned by $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

f) The subspace $\mathrm{Col}(A^T)$ is spanned by the vectors $(1, -1, 2)$ and $(-2, 2, -4)$, but these are scalar multiples of each other, so the row space is a line. The null space can be found via RREF: $\mathrm{rref}(A) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$. The free variables are y and z, and the parametric form is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$. Therefore the null space is a plane in $\mathbf{R}^3$.

g) $\mathrm{Col}(A^T) = \mathrm{Span}\{(1, -1, 2)\}$

h) $\mathrm{Nul}(A) = \mathrm{Span}\{(1, 1, 0), (-2, 0, 1)\}$

i) We consider the matrix $B = \begin{pmatrix} 1 & -2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$, whose column space equals $\mathrm{Nul}(A)$. We find an equation for the column space of B by finding the REF of $\left(\begin{array}{cc|c} 1 & -2 & b_1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{array}\right)$, and finding the equation which makes the system consistent. The REF of this augmented matrix is

$$\left(\begin{array}{cc|r} 1 & -2 & b_1 \\ 0 & 2 & b_2 - b_1 \\ 0 & 0 & b_1 - b_2 + 2b_3 \end{array}\right).$$

The equation that (b_1, b_2, b_3) must satisfy to be in the column space of B (and hence the null space of A) is $b_1 - b_2 + 2b_3 = 0$. In other words, the equation for the plane $\mathrm{Nul}(A)$ is

$$x - y + 2z = 0.$$

j) The coefficients of the equation are $(1, -1, 2)$. This is the same as the vector which spanned $\mathrm{Col}(A^T)$ (you may have gotten a scalar multiple of the vector spanning $\mathrm{Col}(A^T)$ instead.) The means that every vector in the plane is perpendicular to the vector $(1, -1, 2)$, i.e. that the plane has *normal vector* $(1, -1, 2)$. In other words, *the null space is orthogonal to the row space*. We will discuss the orthogonality of subspaces in more detail in Week 7.

5. Linear (in)dependence

a) Since neither vector is a scalar multiple of the other, the two vectors are linearly independent.

b) Any 3 vectors in $\mathbf{R}^2$ must be linearly dependent. To find a dependence, we will compute the null space of the matrix $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & -1 & 2 \end{pmatrix}$. Since the RREF of A is $\begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & 1/2 \end{pmatrix}$, we find that $\begin{pmatrix} -5/2 \\ -1/2 \\ 1 \end{pmatrix}$ a vector in the null space. In other words, $-\frac{5}{2}\binom{1}{1} - \frac{1}{2}\binom{1}{-1} + \binom{3}{2} = 0$ is a linear dependence relation among these three vectors.

c) The dimension is the same as the rank of the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$, since the rank of a matrix equals the dimension of its column space. The rank of this matix is 3 (you could compute its REF, or notice that the transpose A^T is already in REF).

d) Consider any two scalars a, b such that

$$a(u+v) + b(u-v) = 0.$$

We need to show that both of these scalars are in fact equal to 0 - this would show that no linear dependence relations between $u+v$ and $u-v$ are possible.

The first equation implies that $(a+b)u + (b-a)v = a(u+v) + b(u-v) = 0$. Since u and v are linearly independent, this implies that $a+b = 0$ and $b-a = 0$. You can solve these two equations to find $a = 0, b = 0$.

e) The vectors $u+v$, $u+2v-w$, $v-w$ are linearly dependent, since $(u+v)+(v-w) = u + 2v - w$.

f) The matrix $A = \begin{pmatrix} 1 & -1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & -3 & 0 & 1 \\ 1 & -4 & 1 & 1 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 & 1/4 \\ 0 & 1 & 0 & -1/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. The columns of the RREF are dependent: $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} - 4\begin{pmatrix} 1/4 \\ -1/4 \\ -1/4 \\ 0 \end{pmatrix} = 0$. The same dependence relation works for the original vectors (since RREF doesn't change the null space):

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ -2 \\ -3 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 0 \\ 1 \end{pmatrix} - 4\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = 0.$$

6. Bases from an LU decomposition

Suppose that you have an $A = LU$ decomposition, where

$$U = \begin{pmatrix} 1 & -1 & 2 & 3 & 5 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

but you don't know L or A.

a) We can find a basis for Row(A) and Nul(A) - since row operations change Col(A) and Nul(A^T), we can't hope to find them using U. A basis for Row(A) comes from the non-zero rows of U:

$$(1, -1, 2, 3, 5), \ (0, 0, 1, 2, 2), \ (0, 0, 0, 01).$$

To find the null space basis, we finish putting A into RREF – its RREF is $\begin{pmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The parametric equations for $Ax = 0$ are then

$$\begin{aligned} x_1 &= x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \\ x_5 &= 0 \end{aligned}$$

and the parametric vector form is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$. A basis for the null space Nul(A) is given by the two vectors $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

b) We can find all of the dimensions: $\dim \text{Row}(A) = 3$ and $\dim \text{Nul}(A) = 2$ from part a). Since row rank equals column rank equals the number of pivots, $\dim \text{Col}(A) = 3$. Since $\dim \text{Col}(A) + \dim \text{Nul}(A^T) = \#$ of rows $= 4$, we find that $\dim \text{Nul}(A^T) = 1$.