Math 218D Problem Session

Week 7

1. Projection onto a line

a) $b = (1,1),\ v = (1,0)$.

$b_V = \frac{b\cdot v}{v\cdot v}v = 1(1,0) = (1,0)$. Then $b_{V^\perp} = b - b_V = (0,1)$.

b) $b = (0,2),\ v = (1,1)$.

$b_V = \frac{b\cdot v}{v\cdot v}v = \frac{2}{2}v = v = (1,1)$. Then $b_{V^\perp} = b - b_V = (-1,1)$.

c) $b = (1,2,3),\ v = (1,1,-1)$.

$b_V = \frac{b\cdot v}{v\cdot v}v = (0,0,0)$. Then $b_{V^\perp} = b - b_V = b$.

2. Planes and normal vectors

The subspace $V = \mathrm{Span}\{(1,1,2),(1,3,1)\}$ of $\mathbf{R}^3$ is a plane.

a) The matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 3 & 1 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -1/2 \end{pmatrix}$. The null space of A is spanned by $(-5/2, 1/2, 1)$. This is a basis for $V^\perp$.

b) The equation $-\frac{5}{2}x + \frac{1/2}{y} + z = 0$ is true for both $(x,y,z) = (1,1,2)$ and $(x,y,z) = (1,3,1)$.

c) $b = (1,1,1)$. We find $b_{V^\perp}$ first. Note that $V^\perp$ is spanned by $(-5,1,2) = 2(-5/2,1/2,1)$ - this will make the arithmetic a little easier. Then $b_{V^\perp} = \frac{(1,1,1)\cdot(-5,1,2)}{(-5,1,2)\cdot(-5,1,2)}(-5,1,2) = \frac{-2}{30}(-5,1,2) = -\frac{1}{15}(-5,1,2)$.

Then $b_V = b - b_{V^\perp} = (1,1,1) - (-\frac{1}{15}(-5,1,2)) = (10/15, 16/15, 17/15) = (2/3, 16/15, 17/15)$.

3. Projection onto a plane

a) Take the two equations

$$(1,1,1,1)\cdot(\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)) = (1,1,1,1)\cdot(1,-1,-3,-5),$$
$$(1,2,3,4)\cdot(\widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)) = (1,2,3,4)\cdot(1,-1,-3,-5)$$

and compute all dot products (using the distributivity of dot products and addition). We get two equations

$$4\widehat{x}_1 + 10\widehat{x}_2 = -8,$$
$$10\widehat{x}_1 + 30\widehat{x}_2 = -30.$$

b) We solve these two equations by computing the RREF of $\left(\begin{array}{cc|c} 4 & 10 & -8 \\ 10 & 30 & -30 \end{array}\right)$, which is $\left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right)$. Therefore $\widehat{x}_1 = 3, \widehat{x}_3 = -2$, and

$$b_V = 3(1,1,1,1) - 2(1,2,3,4) = (1,-1,-3,-5).$$

(The fact that $b = b_V$ is a bit of a coincidence.)

c) $b_{V^\perp} = b - b_V = 0$, so $b_{V^\perp}$ is orthogonal to V.

d) $A^TA = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$ and $A^Tb = (-8, -30)$. The equation $A^TA\widehat{x} = A^Tb$ is the same as the system of equations

$$4\widehat{x}_1 + 10\widehat{x}_2 = -8,$$

$$10\widehat{x}_1 + 30\widehat{x}_2 = -30$$

from **a)-b)**. The equation $b_V = A\widehat{x}$ is the same as the equation $b_V = \widehat{x}_1(1,1,1,1) + \widehat{x}_2(1,2,3,4)$.

e) The projection matrix is $P = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T$.

We compute the inverse:

$$\begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} = \frac{1}{120-100} \begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix}.$$

Then

$$\begin{aligned} P &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & -3/10 \\ 1/2 & -1/10 \\ 0 & 1/10 \\ -1/2 & 3/10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 7/10 & 2/5 & 1/10 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 \\ 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 1/10 & 2/5 & 7/10 \end{pmatrix}. \end{aligned}$$

This is tedious to compute by hand - I used I computer to multiply these.

f) The matrix $I_4 - P$ equals $\begin{pmatrix} 3/10 & -2/5 & -1/10 & 1/5 \\ -2/5 & 7/10 & -1/5 & -1/10 \\ -1/10 & -1/5 & 7/10 & -2/5 \\ 1/5 & -1/10 & -2/5 & 3/10 \end{pmatrix}$. The first two columns are the same as $(I_4 - P)(1,0,0,0)$ and $(I_4 - P)(0,1,0,0)$. The vectors $(3/10, -2/5, -1/10, 1/5)$ and $(-2/5, 7/10, -1/5, -1/10)$ are then a basis for $V^\perp$. Why? Recall that $P_{V^\perp} = I - P_V$, so both of these vectors are the result of projection onto $V^\perp$, and so are contained in $V^\perp$. Since $V^\perp$ is 2-dimensional (since V was 2-dimensional and $\dim(V) + \dim(V^\perp) = \dim(\mathbf{R}^4) = 4$), to check that these two vectors are a basis, we only need to check that they are not scalar multiples of each other, which is true.

g) We scale the vectors we found in the previous part by 10, to make them simpler, so that $\{(3,-4,-1,2),(-4,7,-2,-1)\}$ are a basis for $V^\perp$. We use these vectors to give equations for V:

$$V = \{(x_1,x_2,x_3,x_4) \in \mathbf{R}^4 : 3x_1-4x_2-x_3+2x_4 = 0 \text{ and } -4x_1+7x_2-2x_3-x_4 = 0\}.$$

Both equations are satisfied by the vectors $(1,1,1,1)$ and $(1,2,3,4)$ — this confirms that we have found correct equations.

4. Projection matrices for lines

a) $L = \text{Span}\{(1,1)\}$ has projection matrix $P_L = \frac{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}{(1,1)\cdot(1,1)} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

b) $L = \text{Span}\{(1,2,3)\}$ has projection matrix

$$P_L = \frac{1}{(1,2,3)\cdot(1,2,3)}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}.$$

c) The line $L = \{(x,y,z) \in \mathbf{R}^3 : 2x+y+z = 0\}^\perp$ is spanned by the vector $(2,1,1)$ (we can see this by looking at the coefficients of the plane equation). Therefore $P_L = \frac{1}{(2,1,1)\cdot(2,1,1)}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}$.

5. **Projection matrices for planes**

a) The projection matrix is $P_V = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T$.

We compute the inverse:

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Then

$$\begin{aligned} P_V &= \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3/2 & -1/2 \\ -1/2 & 1/5 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}^T \\ &= \begin{pmatrix} 1 & -3/10 \\ 1/2 & -1/10 \\ 0 & 1/10 \\ -1/2 & 3/10 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 7/10 & 2/5 & 1/10 & -1/5 \\ 2/5 & 3/10 & 1/5 & 1/10 \\ 1/10 & 1/5 & 3/10 & 2/5 \\ -1/5 & 1/10 & 2/5 & 7/10 \end{pmatrix}. \end{aligned}$$

This is tedious to compute by hand - I used I computer to multiply these.

b) The matrix $I_4 - P_V$ equals $\begin{pmatrix} 3/10 & -2/5 & -1/10 & 1/5 \\ -2/5 & 7/10 & -1/5 & -1/10 \\ -1/10 & -1/5 & 7/10 & -2/5 \\ 1/5 & -1/10 & -2/5 & 3/10 \end{pmatrix}$. The first two columns are the same as $(I_4 - P_V)(1,0,0,0)$ and $(I_4 - P_V)(0,1,0,0)$. The vectors $(3/10, -2/5, -1/10, 1/5)$ and $(-2/5, 7/10, -1/5, -1/10)$ are then a basis for $V^\perp$. Why? Recall that $P_{V^\perp} = I - P_V$, so both of these vectors are the result of projection onto $V^\perp$, and so are contained in $V^\perp$. Since $V^\perp$ is 2-dimensional (since V was 2-dimensional and $\dim(V) + \dim(V^\perp) = \dim(\mathbf{R}^4) = 4$), to check that these two vectors are a basis, we only need to check that they are not scalar multiples of each other, which is true.

c) We scale the vectors we found in the previous part by 10, to make them simpler, so that $\{(3,-4,-1,2), (-4,7,-2,-1)\}$ are a basis for $V^\perp$. We use these vectors to give equations for V:

$$V = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 : 3x_1 - 4x_2 - x_3 + 2x_4 = 0 \text{ and } -4x_1 + 7x_2 - 2x_3 - x_4 = 0\}.$$

Both equations are satisfied by the vectors $(1,1,1,1)$ and $(1,2,3,4)$ — this confirms that we have found correct equations.

6. Some mistakes to avoid

A false "fact": every projection matrix $P = A(A^TA)^{-1}A^T$ equals the identity matrix I.

A false "proof":

$$P = A(A^TA)^{-1}A^T = AA^{-1}(A^T)^{-1}A^T = (AA^{-1})((A^T)^{-1}A^T) = I \cdot I = I.$$

a) The step $(A^TA)^{-1} = A^{-1}A^T$ is incorrect - it only works for square matrices.

b) The proof would be correct when A was a square $n \times n$ matrix with linearly independent columns, i.e. when $\mathrm{Col}(A) = \mathbf{R}^n$. (Note that, since P is the projection onto $V = \mathrm{Col}(A) = \mathbf{R}^n$, it makes sense that $P_V = I_n$.)

Consider the subspace $V = \mathrm{Span}\{(1,1,1,-1),\ (2,1,1,2),\ (3,2,2,1)\}$ in $\mathbb{R}^4$. V is the column space of the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}.$$

c) Since the columns of A are not linearly independent, the matrix A^TA is not invertible.

d) The matrix A has two pivots, in the first and second columns. This means that, if we remove the 3rd column of A, we get a new matrix B with $\mathrm{Col}(B) = V$, and B has full column rank. We can use the projection formula with this matrix $P = B(B^TB)^{-1}B^T$.