MATH 218D-1
PRACTICE FINAL EXAMINATION

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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 180 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. You may bring a 8.5×11**-inch note sheet** covered with anything you want. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 180 minutes, without notes or distractions.

Problem 1.

[30 points]

Consider the following matrix A and its singular value decomposition:

$$A = \begin{pmatrix} 0 & 6 & -3 \\ 5 & 8 & 1 \\ 1 & -8 & 5 \\ 8 & -4 & 10 \end{pmatrix} = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$$

$$\sigma_1 = 3\sqrt{30} \qquad u_1 = \frac{1}{\sqrt{30}} \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix} \qquad v_1 = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

$$\sigma_2 = 3\sqrt{15} \qquad u_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix} \qquad v_2 = \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

a) The rank of A is $r =$ $\boxed{2}$.

b) An orthonormal basis for Col(A) is:

$$\left\{ \frac{1}{\sqrt{30}} \begin{pmatrix} -2 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \frac{1}{\sqrt{15}} \begin{pmatrix} 1 \\ 3 \\ -1 \\ 2 \end{pmatrix} \right\}$$

c) An orthonormal basis for Row(A) is:

$$\left\{ \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \right\}$$

d) An orthonormal basis for Nul(A) is: (hint: cross products)

$$\left\{ \frac{1}{3} \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

e) The *maximum* value of $\|Ax\|$ subject to $\|x\| = 1$ is $\boxed{3\sqrt{30}}$.

f) The *minimum* value of $\|Ax\|$ subject to $\|x\| = 1$ is $\boxed{0}$.

Problem 1, continued

g) Find a left inverse of A (a matrix B such that $BA = I_3$), or explain why no such matrix exists:

No such matrix exists, as A does not have full column rank.

h) The matrix $A^T A$ has an *orthogonal* diagonalization $A^T A = QDQ^T$ for

$$Q = \frac{1}{3}\begin{pmatrix} 1 & 2 & -2 \\ -2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 270 & 0 & 0 \\ 0 & 135 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Problem 2. [25 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \end{pmatrix}.$$

a) Find the singular value decomposition of A in matrix form: $A = U\Sigma V^T$ for

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{18} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 1/2 & 1/2 & -1/\sqrt{2} & 0 \\ 1/2 & -1/2 & 0 & -1/\sqrt{2} \\ 1/2 & -1/2 & 0 & 1/\sqrt{2} \\ 1/2 & 1/2 & 1/\sqrt{2} & 0 \end{pmatrix}$$

Now we change matrices to avoid carry-through error. Consider the following matrix and its singular value decomposition $A = U\Sigma V^T$:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ 2 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & \sqrt{5} & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{10} & -1/\sqrt{3} & -1/\sqrt{15} \\ 1/\sqrt{2} & -1/\sqrt{10} & 1/\sqrt{3} & 1/\sqrt{15} \\ 0 & 2/\sqrt{10} & 1/\sqrt{3} & -2/\sqrt{15} \\ 0 & 2/\sqrt{10} & 0 & 3/\sqrt{15} \end{pmatrix}^T$$

b) Compute the pseudo-inverse of A:

$$A^+ = \frac{1}{15}\begin{pmatrix} 1 & 4 \\ 4 & 1 \\ -3 & 3 \\ -3 & 3 \end{pmatrix}$$

c) Compute the matrix P_{V_1} for projection onto $V_1 = \mathrm{Row}(A)$:

$$P_{V_1} = \frac{1}{5}\begin{pmatrix} 3 & 2 & 1 & 1 \\ 2 & 3 & -1 & -1 \\ 1 & -1 & 2 & 2 \\ 1 & -1 & 2 & 2 \end{pmatrix}$$

d) Compute the matrix P_{V_2} for projection onto $V_2 = \mathrm{Nul}(A)$:

$$P_{V_2} = \frac{1}{5}\begin{pmatrix} 2 & -2 & -1 & -1 \\ -2 & 2 & 1 & 1 \\ -1 & 1 & 3 & -2 \\ -1 & 1 & -2 & 3 \end{pmatrix}$$

e) Compute the orthogonal decomposition of $b = (2,1,4,3)$ with respect to $V_1 = \mathrm{Row}(A)$: $b = b_{V_1} + b_{V_1^\perp}$ for

$$b_{V_1} = \begin{pmatrix} 3 \\ 0 \\ 3 \\ 3 \end{pmatrix} \quad b_{V_1^\perp} = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

f) Compute the orthogonal decomposition of $c = (2,4)$ with respect to $V_3 = \mathrm{Col}(A)$: $c = c_{V_3} + c_{V_3^\perp}$ for

$$c_{V_3} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \quad c_{V_3^\perp} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Problem 3. [20 points]

Consider the positive-definite quadratic form

$$q(x_1, x_2, x_3) = 5x_1^2 + 4x_2^2 + 3x_3^2 + 4x_1x_2 + 4x_2x_3.$$

The minimum value of $\|u\|^2$ subject to $q(u) = 1$ is $\|u\|^2 = 1/7$, and the maximum value of $\|u\|^2$ subject to $q(u) = 1$ is $\|u\|^2 = 1$

a) Find a vector u_1 satisfying $\|u_1\|^2 = 1/7$ and $q(u_1) = 1$.

$$u_1 = \frac{1}{3\sqrt{7}}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

b) Find a vector u_3 satisfying $\|u_3\|^2 = 1$ and $q(u_3) = 1$.

$$u_3 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

c) The maximum value of $\|u\|^2$ subject to $q(u) = 1$ and $u \perp u_1$ is $\|u\|^2 = 1/4$. Find a vector u_2 satisfying $\|u_2\|^2 = 1/4$ and $q(u_2) = 1$ and $u_2 \perp u_1$.

$$u_2 = \frac{1}{6}\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

d) Find a change of coordinates $x = Qy$ such that $q(y)$ is diagonal:

$$x_1 = \frac{1}{3}(2y_1 - 2y_2 + y_3)$$
$$x_2 = \frac{1}{3}(2y_1 + y_2 - 2y_3)$$
$$x_3 = \frac{1}{3}(y_1 + 2y_2 + 2y_3)$$

Problem 4. [20 points]

Consider the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

a) Find the QR decomposition of A.

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \qquad R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{pmatrix}$$

b) Find the least squares solution of $Ax = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ using your answer to **a)**.

$$\widehat{x} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

c) Find the orthogonal projection b_V of $b = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ onto $V = \mathrm{Col}(A)$.

$$b_V = \frac{1}{3}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

d) The singular value decomposition of R is $R = U\Sigma V^T$ for:

$$U = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} \\ \sqrt{2} & \sqrt{3} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{pmatrix} \qquad V = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Find the right singular vectors of A without doing any calculations.

$$v_1 = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad v_2 = \frac{1}{\sqrt{5}}\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

Problem 5.

[15 points]

Consider the matrix

$$A_0 = \begin{pmatrix} 1 & 3 & 2 & 4 & 5 \\ 8 & 2 & 3 & 9 & 8 \\ 3 & 1 & 4 & 5 & 2 \end{pmatrix}$$

whose columns contain five data points of three measurements each.

a) The averages of the rows are

$$\mu_1 = \boxed{3} \qquad \mu_2 = \boxed{6} \qquad \mu_3 = \boxed{3}$$

and the centered data matrix is

$$A = \begin{pmatrix} -2 & 0 & -1 & 1 & 2 \\ 2 & -4 & -3 & 3 & 2 \\ 0 & -2 & 1 & 2 & -1 \end{pmatrix}$$

b) The covariance matrix is

$$S = \frac{1}{4}\begin{pmatrix} 10 & 6 & -1 \\ 6 & 42 & 9 \\ -1 & 9 & 10 \end{pmatrix}$$

The first measurement has variance $s_1^2 = \boxed{\frac{5}{2}}$ and the total variance is $s^2 = \boxed{\frac{31}{2}}$.

The eigenvalues and unit eigenvectors of S are

$$\lambda_1 \approx 11.3 \qquad \lambda_2 \approx 2.74 \qquad \lambda_3 \approx 1.45$$

$$u_1 \approx \begin{pmatrix} 0.156 \\ 0.958 \\ 0.240 \end{pmatrix} \qquad u_2 \approx \begin{pmatrix} -0.792 \\ -0.024 \\ 0.610 \end{pmatrix} \qquad u_3 \approx \begin{pmatrix} -0.590 \\ 0.286 \\ -0.755 \end{pmatrix}.$$

c) In the sense of orthogonal least squares, the best-fit...

line is $L = \text{Span}\left\{\begin{pmatrix} 0.156 \\ 0.958 \\ 0.240 \end{pmatrix}\right\}$ and *plane* is $V = \text{Span}\left\{\begin{pmatrix} 0.156 \\ 0.958 \\ 0.240 \end{pmatrix}, \begin{pmatrix} -0.792 \\ -0.024 \\ 0.610 \end{pmatrix}\right\}$.

The variance along L is $\boxed{11.3}$ and the variance along V is $\boxed{14.04}$.

Problem 6. [25 points]

Fill in the circles of all choices that apply. No justification is necessary.

a) Let A be a 3×3 matrix such that $Ax = (2,1,0)$ does not have a solution. Which of the following are *impossible*?

- ● A is invertible.
- ○ $Ax = (1,2,0)$ does not have a solution.
- ● The solution set of $Ax = (1,2,0)$ is $\{(1,1,0)\}$.
- ● The solution set of $Ax = (4,2,0)$ is a line.
- ● The solution set of $Ax = (2,4,0)$ is a line through the origin.

b) Which of the following sets form a basis for $\mathrm{Nul}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$?

● $\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}\right\}$

○ $\left\{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\-1\\1\\0\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}, \begin{pmatrix}-2\\2\\-2\\2\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}4\\-4\\4\\-4\end{pmatrix}\right\}$

c) Which of the following quadratic forms are positive-definite?

- ○ $q(x_1, x_2, x_3) = x_1^2 - 2x_2^2 + 3x_3^2 + 4x_1x_2 + 2x_1x_3$
- ● $q(x_1, x_2, x_3) = 11x_1^2 + 10x_2^2 + 13x_3^2 + 12x_1x_2 + 6x_1x_3 + 2x_2x_3$
- ○ $q(x_1, x_2, x_3) = (x_1 + x_2 - x_3)^2$
- ● $q(x) = x^T \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} x$

d) Which of the following matrices are diagonalizable over the real numbers?

● $\begin{pmatrix} 1 & 7 & 5 \\ 7 & 2 & 4 \\ 5 & 4 & 9 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ● $\begin{pmatrix} 2 & 7 & 1 & 4 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & -1 & -12 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

○ $\begin{pmatrix} \cos(27^\circ) & -\sin(27^\circ) \\ \sin(27^\circ) & \cos(27^\circ) \end{pmatrix}$ ● $\begin{pmatrix} \text{The matrix} \\ \text{for projection} \\ \text{onto} \end{pmatrix}$ Span $\left\{\begin{pmatrix}1\\3\\2\\7\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\4\end{pmatrix}\right\}$

e) Which of the following phrases do *not* make sense? The letter A refers to a matrix.

- ● The dimension of A is equal to 5.
- ● A is linearly independent.
- ● The subspace of A has dimension 4.
- ○ The solution set of $Ax = (1,2)$ is empty.
- ● The orthogonal complement of A is a plane.

Problem 7.

[25 points]

Give an example of a matrix with each of the following properties, or explain why no such example exists. (No justification is needed if an example does exist.)

All matrices in this problem have real entries.

a) A *non-diagonalizable* 3×3 matrix with eigenvalues 1 and 2.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

b) A *symmetric* matrix whose 1-eigenspace is spanned by $\{(1,2,0),(1,1,0)\}$ and whose 2-eigenspace is spanned by $\{(-1,1,1)\}$.

No such matrix exists: eigenvectors of a symmetric matrix with different eigenvalues are orthogonal.

c) A 2×2, non-diagonal matrix with singular values 1 and 2.

Many answers can be produced by writing down an SVD. For instance,

$$2\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & -1 \\ 2 & 1 \end{pmatrix}.$$

d) A 2×3 matrix that does not have a singular value decomposition.

Every matrix has an SVD.

e) A matrix A satisfying $\dim\left(\mathrm{Row}(A)^{\perp}\right) = 2$ and $\dim\left(\mathrm{Col}(A)^{\perp}\right) = 3$.

Working through the numerics, you need $n - r = 2$ and $m - r = 3$. Taking $r = 1, n = 3$, and $m = 4$, we can use

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Problem 8. [30 points]

True/false problems: **circle** the correct answer. No justification is needed. *All matrices in this problem have real entries.*

a) T **(F)** The singular values of a symmetric matrix are the absolute values of the eigenvalues.

b) **(T)** F If A has linearly independent columns, then A^+A is the identity matrix.

c) **(T)** F A positive-definite matrix has positive numbers on the main diagonal.

d) T **(F)** A diagonalizable 5×5 matrix has 5 different eigenvalues.

e) **(T)** F Every subspace of $\mathbf{R}^n$ has an orthonormal basis.

f) **(T)** F If $V = \{(x, y, z) \in \mathbf{R}^3 \,:\, x = 2y\}$, then $V^\perp$ is a line.

g) T **(F)** If V is a subspace and x is not in V, then the orthogonal projection of x onto V is zero.

h) **(T)** F If V is a subspace and x is not in V, then the orthogonal projection of x onto $V^\perp$ is nonzero.

i) **(T)** F If $\{v_1, \ldots, v_n\}$ is a basis for $\mathbf{R}^5$, then $n = 5$.

j) T **(F)** If λ is an eigenvalue of AA^T then λ is an eigenvalue of A^TA.

Problem 9. [15 points]

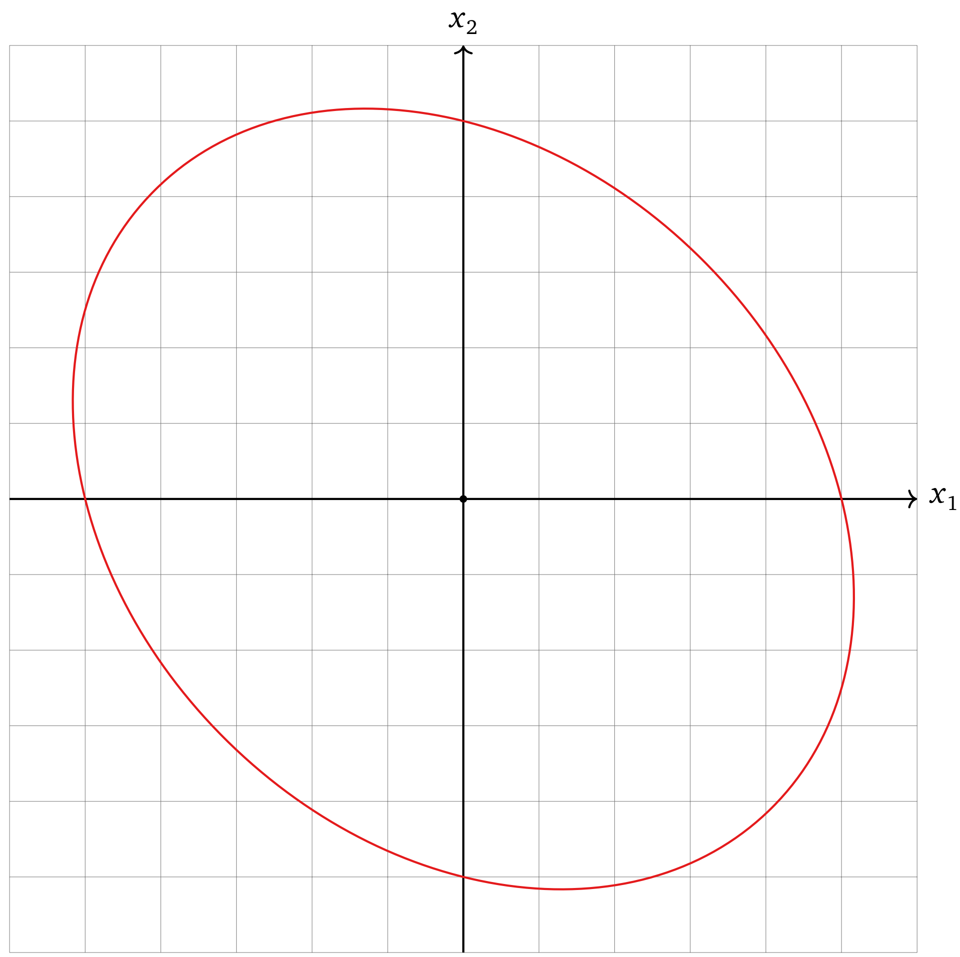
Consider the positive-definite quadratic form

$$q(x_1, x_2) = 4x_1^2 + 4x_2^2 + 2x_1x_2.$$

a) Find a change of variables $x = Qy$ such that $q(y)$ is diagonal, and write the diagonal form $q(y)$:

$$x_1 = \frac{1}{\sqrt{2}}(y_1 - y_2) \qquad q(y_1, y_2) = 5y_1^2 + 3y_2^2$$
$$x_2 = \frac{1}{\sqrt{2}}(y_1 + y_2)$$

b) Draw the set of all points (x_1, x_2) satisfying $q(x_1, x_2) = 1$ on the grid below. Be precise!



grid lines are 0.1 unit apart

Problem 10. [15 points]

A centered data matrix A has 2 rows and 10 columns. Its SVD has the form

$$A = 5u_1v_1^T + 2u_2v_2^T,$$

where u_1, u_2 and v_1, v_2 are the singular vectors. The columns of A and the first left singular vector u_1 are drawn below. *Draw and label*:

a) the best-fit line in the sense of orthogonal least squares;

b) the direction of *smallest* variance;

c) the columns of $5u_1v_1^T$ (drawn as dots).

The variance in the direction of *smallest* variance is $\frac{4}{9}$.