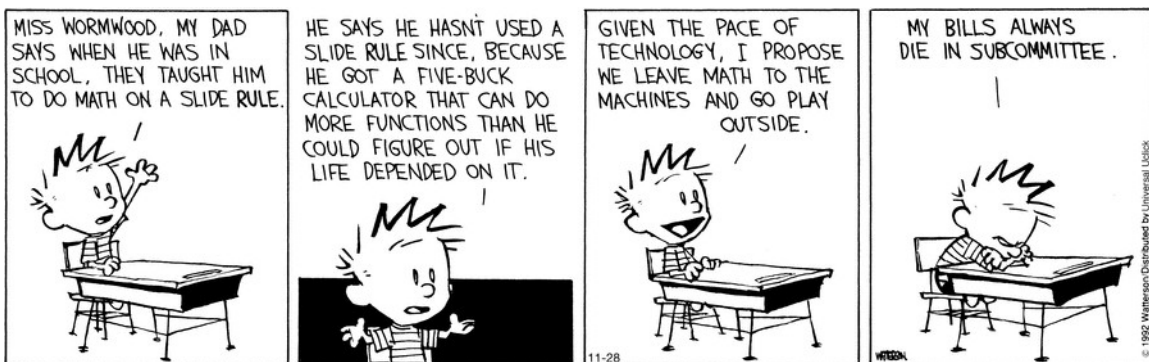


**MATH 218D-1**  
**MAKE-UP MIDTERM EXAMINATION 1**

<b>Name</b>		<b>Duke NetID</b>	
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Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



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## Problem 1.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 4 & 3 & 2 \\ -4 & -5 & 1 \\ 12 & 7 & 3 \end{pmatrix}.$$

a) Find a lower-unitriangular matrix  $L$  and a matrix  $U$  in REF such that  $A = LU$ .

$$L = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \quad U = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

b) Use the LU decomposition you found in a) to solve the equation  $Ax = (3, 2, 8)$ .

$$x = \begin{pmatrix} \\ \\ \end{pmatrix}$$

c) Briefly explain why one would want to use an LU decomposition to solve  $Ax = b$ .

d) Express  $L$  as a product of three elementary matrices.

$$L = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

[Scratch work for Problem 1]

## Problem 2.

[25 points]

Consider the system of equations

$$\begin{aligned}x_1 + x_3 + x_4 &= 3 \\-x_1 + x_2 + 3x_3 - x_4 &= -2 \\x_1 + 2x_2 + 9x_3 + 2x_4 &= 5.\end{aligned}$$

a) Express this system of equations as a matrix equation  $Ax = b$ :

$$\begin{pmatrix} & & & & \\ & & & & \\ & & & & \end{pmatrix} x = \begin{pmatrix} \\ \\ \end{pmatrix}$$

b) Perform Gauss–Jordan elimination on the augmented matrix  $(A | b)$  to produce a matrix in reduced row echelon form. Please write all row operations that you perform.

$$\left( \begin{array}{cccc|c} & & & & \\ & & & & \\ & & & & \end{array} \right)$$

c) The free variable(s) are  and the rank of  $A$  is .

[Scratch work for Problem 2]

(Problem 2, continued)

d) Write the solution set of this system of equations in parametric vector form.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{pmatrix} +$$

e) The solution set is a (circle one)  $\begin{pmatrix} \text{line} \\ \text{plane} \\ \text{space} \end{pmatrix}$  in (fill in the blank)  $\mathbf{R}^{\boxed{\phantom{000}}}$ .

f) Find a basis for  $\text{Nul}(A)$ .

$$\left\{ \begin{pmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{pmatrix} \right\}$$

g) Write down any *nontrivial* solution of  $Ax = 0$ .

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \phantom{x} \\ \phantom{x} \\ \phantom{x} \\ \phantom{x} \end{pmatrix}$$

[Scratch work for Problem 2]



### Problem 3.

[20 points]

Consider the vectors

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} \quad v_4 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}.$$

a) Find a linear relation among  $v_1, v_2, v_3, v_4$ .

0 =
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b) The set  $\{v_1, v_2, v_3, v_4\}$  is (circle one)  $\left( \begin{array}{l} \text{linearly dependent} \\ \text{linearly independent} \end{array} \right)$ .

c) Which of the following vectors is in  $\text{Span}\{v_1, v_2, v_3, v_4\}$ ?

(Circle all that apply.)

$$u = \begin{pmatrix} 2 \\ -6 \\ -5 \end{pmatrix} \quad w = \begin{pmatrix} 2 \\ -6 \\ 5 \end{pmatrix}$$

d) Express  $(3, -2, 5)$  as a linear combination of  $v_1, v_2, v_3, v_4$  in two different ways.

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \boxed{\phantom{00}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} = \boxed{\phantom{00}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 1 \\ 3 \\ 9 \end{pmatrix} + \boxed{\phantom{00}} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

e) Which of the following sets form a basis for  $\text{Span}\{v_1, v_2, v_3, v_4\}$ ?

(Circle all that apply.)

$$\{v_1, v_2\} \quad \{v_1, v_2, v_3\} \quad \{v_1, v_2, v_4\}$$

$$\{v_3, v_4\} \quad \{v_2, v_3, v_4\} \quad \{v_2, v_4\}$$

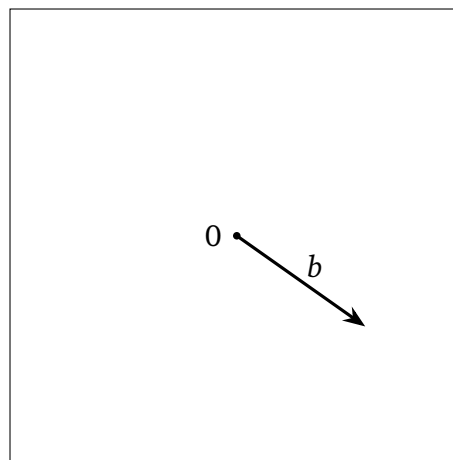
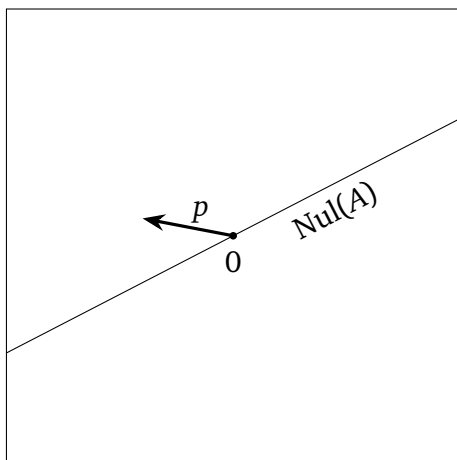
f)  $\dim(\text{Span}\{v_1, v_2, v_3, v_4\}) = \boxed{\phantom{00}}$

[Scratch work for Problem 3]

### Problem 4.

[15 points]

For a certain  $2 \times 2$  matrix  $A$  and vector  $p \in \mathbf{R}^2$ , the vector  $p$  and the null space of  $A$  is drawn in the picture on the left, and the vector  $b = Ap$  is drawn in the picture on the right.



- a) Draw the solution set of  $Ax = b$  in the picture on the left.
- b) Draw  $\text{Col}(A)$  in the picture on the right.
- c) Draw a vector  $b'$  in the picture on the right that makes the system  $Ax = b'$  inconsistent.

[Scratch work for Problem 4]

## Problem 5.

[10 points]

For each of the following subsets,

- determine if the subset is a subspace.

If it is not a subspace,

- explain why not.

If it is a subspace,

- express it as the null space or the column space of a matrix, and
- find a basis.

a) 
$$V_1 = \left\{ \begin{pmatrix} a \\ a+1 \end{pmatrix} : a \in \mathbf{R} \right\}$$

b) 
$$V_2 = \left\{ \begin{pmatrix} a+b \\ 0 \\ 2a-2b \end{pmatrix} : a, b, c \in \mathbf{R} \right\}$$

c) 
$$V_3 = \text{the } y\text{-axis in the } xy\text{-plane}$$

d) 
$$V_4 = \text{all vectors } \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ such that } x + 2 \leq y$$

[Scratch work for Problem 5]



[Scratch work for Problem 6]