MATH 218D-1
MIDTERM EXAMINATION 2

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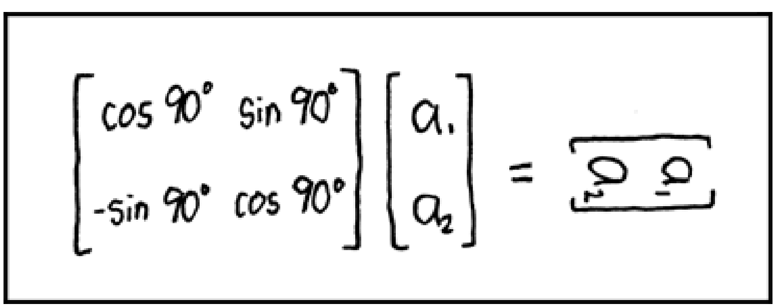
Please **read all instructions** carefully before beginning.

- Do not open this test booklet until you are directed to do so.
- You have 75 minutes to complete this exam.
- If you finish early, go back and check your work.
- The graders will only see the work on the **printed pages** (front and back). You may use other scratch paper, but the graders will not see anything written there.
- You may use a **four-function calculator** for doing arithmetic, but you should not need one. All other materials and aids are strictly prohibited.
- For full credit you must **show your work** so that your reasoning is clear, unless otherwise indicated.
- Do not spend too much time on any one problem. Read them all through first and attack them in an order that allows you to make the most progress.
- Good luck!



[Hint: this is a joke.]

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Problem 1. [20 points]

Consider the subspace

a) Find an orthogonal basis for

$$\operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 3 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

$$\left\{ \qquad\qquad\qquad\qquad \right\}$$

[Scratch work for Problem 1]

(Problem 1, continued)

Now we change subspaces to avoid carry-through error. Consider the subspace

$$V = \text{Span}\left\{\begin{pmatrix}1\\2\\-2\end{pmatrix}, \begin{pmatrix}-2\\2\\1\end{pmatrix}\right\},$$

and note that the spanning vectors are *orthogonal*.

b) Compute the orthogonal projection of $b = (-7, 4, -4)$ onto V.

$$b_V = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

c) Compute the matrix P_V for projection onto V.

$$P_V = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

d) Compute the matrix $P_{V^\perp}$ for projection onto $V^\perp$.

$$P_{V^\perp} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

e) Find a basis of $\text{Nul}(P_V)$.

$$\left\{ \qquad \right\}$$

[Scratch work for Problem 1]

Problem 2. [15 points]

In this problem we will consider the best-fit line $y = Cx + D$ through the data points

$$\begin{pmatrix}1\\b_1\end{pmatrix}, \quad \begin{pmatrix}2\\b_2\end{pmatrix}, \quad \begin{pmatrix}3\\b_3\end{pmatrix}, \quad \begin{pmatrix}4\\b_4\end{pmatrix}.$$

a) The line $y = Cx + D$ passes through all four points if and only if the matrix equation

$$\begin{pmatrix} & \\ & \\ & \\ & \end{pmatrix}\begin{pmatrix}C\\D\end{pmatrix} = \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix}$$

is satisfied (fill in the blank).

Let A be the coefficient matrix in the previous problem. In the QR decomposition of A, the matrix Q is

$$Q = \begin{pmatrix} 1/\sqrt{30} & 2/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{6} \\ 3/\sqrt{30} & 0 \\ 4/\sqrt{30} & -1/\sqrt{6} \end{pmatrix}.$$

b) Explain why $R = Q^T A$, and compute R.

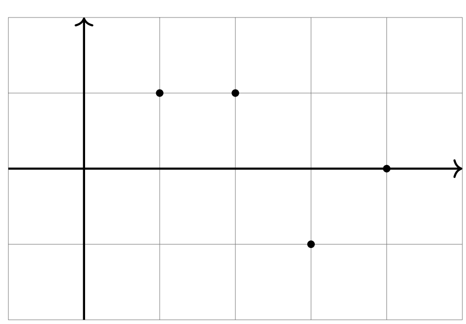
$$R = \begin{pmatrix} & \\ & \end{pmatrix}$$

c) Use the QR decomposition to find the best-fit line through the data points

$$\begin{pmatrix}1\\1\end{pmatrix}, \quad \begin{pmatrix}2\\1\end{pmatrix}, \quad \begin{pmatrix}3\\-1\end{pmatrix}, \quad \begin{pmatrix}4\\0\end{pmatrix}.$$

$y = \square\, x + \square$

d) Graph the line you found in **c)** below. Explain which quantity was minimized in terms of the graph.



[Scratch work for Problem 2]

Problem 3.

[15 points]

Consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -2 \\ -1 & -1 & 1 & 1 \\ 3 & 3 & -1 & -5 \end{pmatrix}.$$

a) Compute bases of all four fundamental subspaces of A.

$\mathrm{Nul}(A)$: $\left\{ \qquad \right\}$ $\quad$ $\mathrm{Col}(A)$: $\left\{ \qquad \right\}$

$\mathrm{Row}(A)$: $\left\{ \qquad \right\}$ $\quad$ $\mathrm{Nul}(A^T)$: $\left\{ \qquad \right\}$

b) Compute the orthogonal decomposition of $(0, 3, 3)$ with respect to $V = \mathrm{Col}(A)$.

$$\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} =$$

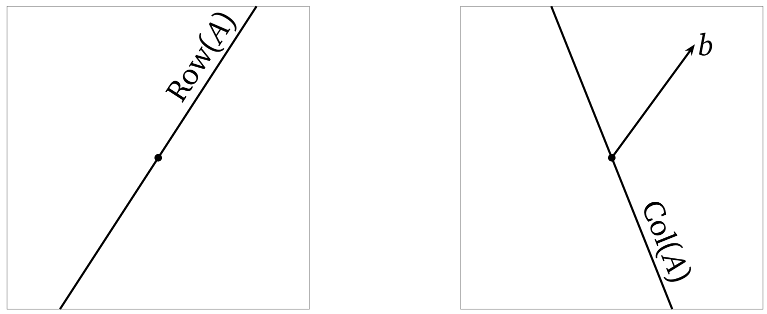
c) Compute the distance from $(0, 3, 3)$ to $\mathrm{Col}(A)$.

[Scratch work for Problem 3]

Problem 4.

[10 points]

For a certain 2×2 matrix A, the row space of A is drawn in the picture on the left, and the column space is drawn in the picture on the right.



a) $\text{rank}(A) = \square$.

b) Draw $\text{Nul}(A)$ in the picture on the left.

c) Draw $\text{Nul}(A^T)$ in the picture on the right.

d) If $V = \text{Col}(A)$ and b is the vector in the picture on the right, draw and label the vectors b_V and $b_{V^\perp}$.

[Scratch work for Problem 4]

Problem 5. [20 points]

a) Find a matrix whose null space is $\text{Span}\{(1,1,1)\}$.

b) For which value(s) of k, if any, do the following vectors *not* form a basis of $\mathbf{R}^4$?

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ -6 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -8 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \\ k \end{pmatrix} \right\}$$

c) Which of the following properties of a matrix are not changed by row operations? (Fill in the bubbles of all that apply)

- ◯ The rank
- ◯ The column space
- ◯ The null space
- ◯ The row space
- ◯ The left null space
- ◯ The determinant
- ◯ The reduced row echelon form

d) If $\det(A) = 2$ and $\det(B) = 3$, compute the following determinants:

$\det(A^2) = \square$ $\det(AB^T) = \square$ $\det(BA^kB^{-1}) = \square$

(Here A and B are square matrices of the same size and k is a whole number.)

[Scratch work for Problem 5]

Problem 6. [20 points]

Give examples of matrices with the following properties. If no such matrix exists, explain why.

a) A 3×2 matrix A such that $Ax = (1,2,3)$ has more than one least-squares solution.

b) A matrix A in RREF satisfying $\dim \mathrm{Row}(A) = 2$ and $\dim \mathrm{Nul}(A) = 3$.

c) A matrix Q with orthonormal columns, such that $\det(QQ^T) = 0$.

d) A matrix A whose column space $V = \mathrm{Col}(A)$ is a plane in $\mathbf{R}^3$, such that $\mathrm{rank}(P_V) = 1$.

[Scratch work for Problem 6]