Math 218D-1: Homework #1

due Wednesday, September 7, at 11:59pm

1. Consider the vectors

$$u = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 8 \\ -6 \\ -2 \end{pmatrix}.$$

a) Compute $u+v+w$ and $u+2v-w$.

b) Find numbers x and y such that $w = xu + yv$.

c) Explain why every linear combination of u, v, w is also a linear combination of u and v only.

d) The sum of the coordinates of any linear combination of u, v, w is equal to ____?

e) Find a vector in $\mathbf{R}^3$ that is *not* a linear combination of u, v, w.

2. Find two *different* triples (x, y, z) such that

$$x\begin{pmatrix} 1 \\ 2 \end{pmatrix} + y\begin{pmatrix} 1 \\ -2 \end{pmatrix} + z\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

How many such triples are there?

3. Decide if each statement is true or false, and explain why.

a) The vector $\frac{1}{2}v$ is a linear combination of v and w.

b) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

c) If v, w are two vectors in $\mathbf{R}^2$, then any other vector b in $\mathbf{R}^2$ is a linear combination of v and w.

4. Suppose that v and w are *unit vectors*: that is, $v \cdot v = 1$ and $w \cdot w = 1$. Compute the following dot products (your answers will be actual numbers):

a) $v \cdot (-v)$ **b)** $(v+w)\cdot(v-w)$ **c)** $(v+2w)\cdot(v-2w)$.

5. Two vectors v and w are *orthogonal* if $v \cdot w = 0$, and they are *parallel* if one is a scalar multiple of the other. A *unit vector* is a vector v with $v \cdot v = 1$.

Decide if each statement is true or false, and explain why.

a) If $u = (1, 1, 1)$ is orthogonal to v and to w, then v is parallel to w.

b) If u is orthogonal to $v+w$ and to $v-w$, then u is orthogonal to v and w.

c) If u and v are orthogonal unit vectors then $(u-v)\cdot(u-v) = 2$.

d) If $u \cdot u + v \cdot v = (u+v) \cdot (u+v)$, then u and v are orthogonal.

6. Find nonzero vectors v and w that are orthogonal to $(1,1,1)$ and to each other.

7. Compute the following matrix-vector products using *both* the by-row and by-column methods. If the product is not defined, explain why.

$$\begin{pmatrix} 2 \\ 5 \end{pmatrix}\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 7 & 2 & 4 \\ 3 & -3 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 7 & 4 \\ -2 & 2 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 6 & -1 \end{pmatrix}\begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 2 & 6 & -1 \end{pmatrix}$$

8. Suppose that $u = (x, y, z)$ and $v = (a, b, c)$ are vectors satisfying $2u + 3v = 0$. Find a nonzero vector w in $\mathbf{R}^2$ such that

$$\begin{pmatrix} x & a \\ y & b \\ z & c \end{pmatrix} w = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

9. Consider the matrices

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -4 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & -1 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$$

$$D = \begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix} \qquad E = \begin{pmatrix} -3 & 5 \end{pmatrix}.$$

Compute the following expressions. If the result is not defined, explain why.

a) $-3A$ **b)** $B - 3A$ **c)** AC **d)** B^2

e) $A + 2B$ **f)** $C - E$ **g)** EB **h)** D^2

10. Compute the product

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

in three ways:

a) Using the column form and the "by columns" method on each column.

b) Using the column form and the "by rows" method on each column.

c) Using the outer product form.

11. Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ -1 & h \end{pmatrix}.$$

What value(s) of h, if any, will make $AB = BA$?

12. Consider the matrices

$$A = \begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & -8 \\ 5 & 8 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Verify that $AC = BC$ and yet $A \neq B$.

13. For the following matrices A and B, compute AB, A^T, B^T, B^TA^T, and $(AB)^T$. Which of these matrices are equal and why? Why can't you compute A^TB^T?

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}.$$

14. Decide if each statement is true or false, and explain why.

a) If A and B are symmetric of the same size, then AB is symmetric.

b) If A is symmetric, then A^3 is symmetric.

c) If A is any matrix, then A^TA is symmetric.

15. In the table below, a linear system is expressed as a system of equations, as a matrix equation, or as an augmented matrix. Fill in the blank entries.

System of Equations	**Matrix Equation**	**Augmented Matrix**
$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 9 \\ -x_1 + 4x_3 &= 2 \end{aligned}$		
	$\begin{pmatrix} 3 & -5 \\ 2 & 4 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$	
		$\left(\begin{array}{cccc\|c} 1 & 0 & 1 & 1 & 2 \\ 0 & 3 & -1 & -2 & 4 \\ 1 & -3 & -4 & -3 & 2 \\ 6 & 5 & -1 & -8 & 1 \end{array}\right)$

16. Which of the following matrices are not in row echelon form? Why not?

$$\begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 1 & 0 \\ 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 9 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \\ 2 \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 4 \\ 0 & 0 \end{pmatrix}$$

17. Consider the following system of equations:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 1 \\ -2x_1 + 5x_2 + 5x_3 &= 2 \\ 3x_1 - 7x_2 - 7x_3 &= 2. \end{aligned}$$

a) Use row operations to eliminate x_1 from all but the first equation.

b) Use row operations to modify the system so that x_2 only appears in the first and second equations (and x_1 still only appears in the first).

c) Solve for x_3, then for x_2, then for x_1. What is the solution?

18. The matrix below can be transformed into row echelon form using exactly two row operations. What are they?

$$\begin{pmatrix} 2 & 4 & -2 & 4 \\ -1 & -2 & 1 & -2 \\ 0 & 2 & 0 & 3 \end{pmatrix}$$

19. Find values of a and b such that the following system has **a)** zero, **b)** exactly one, and **c)** infinitely many solutions.

$$\begin{aligned} 2x + ay &= 4 \\ x - y &= b \end{aligned}$$

20. Give examples of matrices A in *row echelon form* for which the number of solutions of $Ax = b$ is:

a) 0 or 1, depending on b

b) ∞ for every b

c) 0 or ∞, depending on b

d) 1 for every b.

Is there a square matrix satisfying **b)**? Why or why not?