Math 218D-1: Homework #14

due Wednesday, December 7, at 11:59pm

1. For each matrix A, find the singular value decomposition in the outer product form

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T.$$

a) $\begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$ **b)** $\begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ **c)** $\begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$

d) $\begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$ **e)** $\begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$

2. Consider the matrix

$$A = \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix}$$

of Problem 1(a). Let σ_1, σ_2 be the singular values of A. Find *all* singular value decompositions $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$.

3. Let A be a matrix with nonzero orthogonal columns $w_1, \ldots, w_n$ of lengths $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$, respectively. Find the SVD of A in outer product form.

4. Let S be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity). Order the eigenvalues so that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_r| > 0 = \lambda_{r+1} = \cdots = \lambda_n$. Let $\{v_1, \ldots, v_n\}$ be an orthonormal eigenbasis, where v_i has eigenvalue λ_i.

a) Show that the singular values of S are $|\lambda_1|, \ldots, |\lambda_r|$. In particular, $\mathrm{rank}(S) = r$.

b) Find the singular value decomposition of S in outer product form, in terms of the λ_i and the v_i.

5. **a)** Show that all singular values of an orthogonal matrix are equal to 1.

b) Let A be an $m \times n$ matrix, let Q_1 be an $m \times m$ orthogonal matrix, and let Q_2 be an $n \times n$ orthogonal matrix. Show that A has the *same singular values* as $Q_1 A Q_2$. [**Hint:** Use HW10#6.]

Remark: This fact is heavily exploited when numerically computing the SVD: a complicated matrix is simplified by multiplying on the left and right by simple orthogonal matrices.

6. Let A be a matrix of full column rank and let $A = QR$ be the QR decomposition of A.

a) Show that A and R have the same singular values $\sigma_1, \ldots, \sigma_r$ and the same right singular vectors $v_1, \ldots, v_r$.

b) What is the relationship between the left singular vectors of A and R?

7. Let A be a matrix with first singular value σ_1 and first right singular vector v_1. Recall that the *matrix norm* of A is the maximum value of $\|Ax\|$ subject to $\|x\| = 1$, and is denoted $\|A\|$.

a) Show that $\|Ax\|$ is maximized at $x = v_1$ (subject to $\|x\| = 1$), with maximum value σ_1.

b) Suppose now that A is square and λ is an eigenvalue of A. Show that $|\lambda| \leq \sigma_1$. (You may assume λ is real, although it is also true for complex eigenvalues.)

This shows that *the largest singular value is at least as big as the largest eigenvalue.*

8. **a)** Find the eigenvalues and singular values of

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

b) Find the (real and complex) eigenvalues and singular values of

$$A' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0.0001 & 0 & 0 & 0 \end{pmatrix}.$$

c) Note that A is very close to A' numerically. Were the eigenvalues of A close to the eigenvalues of A'? What about the singular values?

This problem is meant to illustrate the fact that *eigenvalues are numerically unstable* but *singular values are numerically stable*. This is another advantage of the SVD.

9. Decide if each statement is true or false, and explain why.

a) The left singular vectors of A are eigenvectors of A^TA and the right singular vectors are eigenvectors of AA^T.

b) For any matrix A, the matrices AA^T and A^TA have the same nonzero eigenvalues.

c) If S is symmetric, then the nonzero eigenvalues of S are its singular values.

d) If A does not have full column rank, then 0 is a singular value of A.

e) Suppose that A is invertible with singular values $\sigma_1, \ldots, \sigma_n$. Then for $c \geq 0$, the singular values of $A + cI_n$ are $\sigma_1 + c, \ldots, \sigma_n + c$.

f) The right singular vectors of A are orthogonal to Nul(A).

10. For each matrix A of Problem 1:

$$\textbf{a)}\ \begin{pmatrix} 8 & 4 \\ 1 & 13 \end{pmatrix} \qquad \textbf{b)}\ \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad \textbf{c)}\ \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

$$\textbf{d)}\ \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix} \qquad \textbf{e)}\ \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

find the singular value decomposition in the matrix form

$$A = U\Sigma V^T.$$

11. For each matrix A of Problem 10, write down orthonormal bases for all four fundamental subspaces. (This can be read off from your answers to Problem 10.)

12.

a) Let A be an invertible $n \times n$ matrix. Show that the product of the singular values of A equals the absolute value of the product of the (real and complex) eigenvalues of A (counted with algebraic multiplicity).

[**Hint:** Both equal $|\det(A)|$. What is $\det(A^T A)$?]

b) Find an example of a 2×2 matrix A with distinct positive eigenvalues that are not equal to any of the singular values of A.

[**Hint:** One of the matrices in Problem 1 works.]

13. Let A be a square, invertible matrix with singular values $\sigma_1, \ldots, \sigma_n$.

a) Show that A^{-1} has the same singular vectors as A^T, with singular values $\sigma_n^{-1} \geq \cdots \geq \sigma_1^{-1}$. [**Hint:** What is A^+?]

b) Let λ be an eigenvalue of A. Use Problem 7(b) and **a)** to show that $\sigma_n \leq |\lambda|$.

It follows that the absolute values of all eigenvalues of A are contained in the interval $[\sigma_n, \sigma_1]$. Compare Problem 12.

14. Let S be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $S = QDQ^T$ be an orthogonal diagonalization of S, where D has diagonal entries $\lambda_1, \ldots, \lambda_n$.

Show that $S = QDQ^T$ is a singular value decomposition if and only if S is positive-semidefinite. [See Problem 4.]

15. Compute the pseudoinverse of each matrix of Problem 10.

16. **a)** Find a *left inverse* of the matrix

$$A = \begin{pmatrix} -3 & 11 \\ 10 & -2 \\ 1 & 5 \\ -4 & 6 \end{pmatrix}$$

from Problem 10(c). (This is a matrix B such that BA is the identity.)

b) Find a *right inverse* of the matrix

$$A = \begin{pmatrix} 9 & 7 & 10 & 8 \\ -13 & 1 & 5 & -6 \end{pmatrix}$$

from Problem 10(d). (This is a matrix B such that AB is the identity.)

c) Explain why the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$$

from Problem 10(b) does not admit a left inverse or a right inverse.

17. Consider the matrix

$$A = \begin{pmatrix} 3 & 7 & 1 & 5 \\ 3 & 1 & 7 & 5 \\ 6 & 2 & 2 & -2 \end{pmatrix}$$

of Problem 15(e). Find the matrix P_V for projection onto $V = \text{Row}(A)$ in two ways:

a) Multiply out $P_V = A^+A$.

b) In Problem 11 you found $\text{Nul}(A) = \text{Span}\{v\}$ for $v = (1, -1, -1, 1)$. Compute $P_{V^\perp} = vv^T / v \cdot v$ and $P_V = I_4 - P_{V^\perp}$.

Your answers to **a)** and **b)** should be the same, of course!

18. Let A be a matrix and let A^+ be its pseudoinverse. Match the subspaces on the left to the subspaces on the right:

$\text{Col}(A)$	$\text{Col}(A^+)$
$\text{Nul}(A)$	$\text{Nul}(A^+)$
$\text{Row}(A)$	$\text{Row}(A^+)$
$\text{Nul}(A^T)$	$\text{Nul}((A^+)^T)$

What is the rank of A^+?

19. What is the pseudoinverse of the $m \times n$ zero matrix?

20. Consider the matrix $A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix}$ of Problem 15(b).

a) Find all least-squares solutions of $Ax = \binom{3}{1}$ in parametric vector form.

b) Find the shortest least-squares solution $\widehat{x} = A^{+}\binom{3}{1}$.

c) Draw your answers to **a)** and **b)** on the grid below.