Math 218D-1: Homework #7

due Wednesday, October 19, at 11:59pm

1. Find all least-squares solutions $\widehat{x}$ of each of the following systems of equations $Ax = b$, and compute the projection b_V of b onto $V = \mathrm{Col}(A)$ and the minimum value of $\|A\widehat{x} - b\|$.

a) $\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$

b) $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 0 \\ 2 & 2 & -1 \\ 4 & 3 & 0 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 7 \end{pmatrix}$

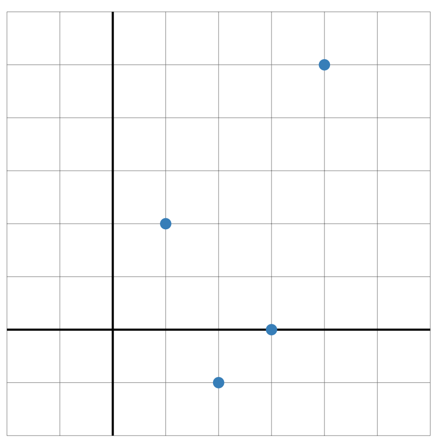
c) $\begin{pmatrix} 2 & 2 & -1 \\ -4 & -5 & 5 \\ 6 & 1 & 12 \end{pmatrix} x = \begin{pmatrix} -6 \\ -24 \\ -3 \end{pmatrix}$

d) $\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ 3 & 1 \end{pmatrix} x = \begin{pmatrix} 9 \\ 7 \\ 7 \end{pmatrix}$

2. Consider the data points

$$p_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

a) Find the best-fit line $y = Cx + D$ through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What is the minimum value of $\sum_{i=1}^{4}(b_i - y(a_i))^2$? How do you know?

d) Verify that the vector

$$\big(2 - y(1),\ -1 - y(2),\ 0 - y(3),\ 5 - y(4)\big)$$

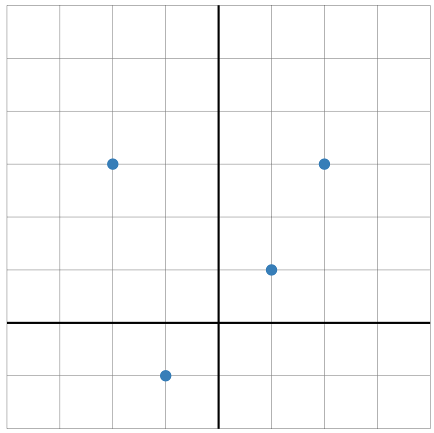
is orthogonal to $(1,2,3,4)$ and $(1,1,1,1)$, and explain why this is necessary.

e) Find the best-fit *horizontal* line $y = D$ through these four points. Verify that D is the average of the y-values of the data points p_1, p_2, p_3, p_4.

3. Consider the data points

$$p_1 = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \qquad p_2 = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad p_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p_4 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

a) Find the best-fit parabola $y = Cx^2 + Dx + E$ through these four points, and draw it on the grid below.



b) For each data point $p_i = (a_i, b_i)$, draw the error bar from $(a_i, y(a_i))$ to (a_i, b_i).

c) What is the minimum value of $\sum_{i=1}^{4}(b_i - y(a_i))^2$? How do you know?

4. Consider the following data points:

$$p_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad p_2 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad p_3 = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} \qquad p_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

a) Find the best-fit plane $z = Cx + Dy + E$ through these four points.

b) Interpret the minimized quantity in the situation of this problem.

5. Consider the data points $p_1, \ldots, p_8$:

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 1.5 \end{pmatrix}, \begin{pmatrix} 2.5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -.5 \\ -1 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -2.5 \\ 2 \end{pmatrix}, \begin{pmatrix} -1.5 \\ 3.5 \end{pmatrix}.$$

a) Find the best-fit *ellipse*

$$x^2 + By^2 + Cxy + Dx + Ey + F = 0$$

through these data points.

b) Interpret the minimized quantity in the situation of this problem.
[**Hint:** you can't see it on the graph above, but you can see it on this demo.]

In this problem, I recommend using SymPy (in the Sage cell on the course webpage) or another computer algebra system to do the computations. To solve a normal equation $A^TAx = A^T(1,2,3)$, you would use something like

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(A.transpose()*A).solve(A.transpose()*Matrix([1,2,3]))
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Remark: Carl Friedrich Gauss (1777–1865), arguably the greatest mathematician since antiquity, kept food on the table by doing astronomical calculations. He invented much of the linear algebra you are learning in order to compute the trajectories of celestial bodies. Essentially performing the calculations in this problem, he correctly predicted the (elliptical) orbit of the asteroid Ceres as it passed behind the sun in 1801.

6. Suppose that $\widehat{x}$ is a vector such that $A\widehat{x} = (1,1,-1,-1)$. Explain why $\widehat{x}$ is not a least-squares solution of $Ax = (1,1,1,1)$.

7. Decide if each statement is true or false, and explain why.

a) A least-squares solution $\widehat{x}$ of $Ax = b$ is a solution of $A\widehat{x} = b_{\mathrm{Col}(A)}$.

b) Any solution of $A^T A\widehat{x} = A^T b$ is a least-squares solution of $Ax = b$.

c) If A has full column rank, then $Ax = b$ has exactly one least-squares solution for every b.

d) If $Ax = b$ has at least one least-squares solution for every b, then A has full row rank.