

Orthogonal Bases

Last time: we found the best approximate soln of $Ax=b$ using least squares.

Now we turn to **computational** considerations.

The goal is the QR decomposition.

LU makes solving $Ax=b$ fast | QR makes least- $\square$ solving $Ax=b$ fast.

("fast" means: no elimination necessary)

The basic idea is that projections are easier when you have a basis of **orthogonal** vectors.

Def: A set of **nonzero** vectors $\{u_1, \dots, u_n\}$ is:

(1) **orthogonal** if $u_i \cdot u_j = 0$ for $i \neq j$

(2) **orthonormal** if they're orthogonal **and** $u_i \cdot u_i = 1$ for all i (unit vectors).

Let $Q = \begin{pmatrix} | & & | \\ u_1 & \dots & u_n \\ | & & | \end{pmatrix}$, so $Q^T Q = \begin{pmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & \dots \\ u_2 \cdot u_1 & u_2 \cdot u_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$.

(1) $\{u_1, \dots, u_n\}$ is **orthogonal** $\iff Q^T Q$ is **diagonal** (& invertible)

$\uparrow$ all nonzero entries are on the diagonal

(2) $\{u_1, \dots, u_n\}$ is **orthonormal** $\iff Q^T Q = I_n$

Q: Does $Q^T Q = I_n$ mean $Q^T = Q^{-1}$?

→ only if Q is square

Eg: $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $u_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

(1) $u_1 \cdot u_2 = 0 \Rightarrow \{u_1, u_2\}$ is orthogonal

(2) $u_1 \cdot u_1 = 4 = u_2 \cdot u_2 \Rightarrow \{u_1, u_2\}$ is not orthonormal

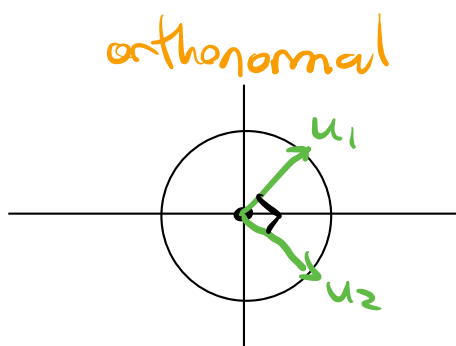
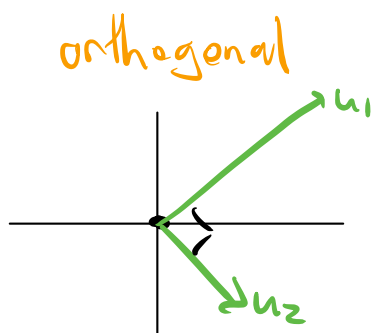
$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \rightsquigarrow Q^T Q = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$$

NB: Given an orthogonal set $\{u_1, \dots, u_n\}$ you can make it orthonormal by dividing by lengths:

$$v_i = \frac{u_i}{\|u_i\|} \rightsquigarrow \{v_1, \dots, v_n\} \text{ is orthonormal}$$

Eg: $v_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $v_2 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ $\{v_1, v_2\}$ is o.n.

Picture in $\mathbb{R}^2$:



Fact:

Let $\{u_1, \dots, u_n\}$ be an orthogonal set and let $Q = (u_1 \dots u_n)$. Then $\{u_1, \dots, u_n\}$ is linearly independent. Equivalently, Q has full column rank.

This means $\{u_1, \dots, u_n\}$ is a basis for $\text{Span}\{u_1, \dots, u_n\}$

Proof: Say $x_1 u_1 + \dots + x_n u_n = 0$. Take $(\cdot) \cdot u_1$:

$$\begin{aligned} 0 &= 0 \cdot u_1 = (x_1 u_1 + \dots + x_n u_n) \cdot u_1 \\ &= x_1 u_1 \cdot u_1 + \cancel{x_2 u_2 \cdot u_1} + \dots + \cancel{x_n u_n \cdot u_1} \\ &= x_1 \|u_1\|^2 \Rightarrow x_1 = 0 \end{aligned}$$

Do the same for $u_2, u_3, \dots$



Projection formula:

Let $\{u_1, \dots, u_n\}$ be an orthogonal set and let $V = \text{Span}\{u_1, \dots, u_n\}$. For any vector b ,

$$b_V = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} u_2 + \dots + \frac{b \cdot u_n}{u_n \cdot u_n} u_n$$

[demo]

NB: faster than $A^T A \tilde{x} = A^T b$: no elimination necessary!

NB: $n=1 \leadsto$ get projection onto a line $b_v = \frac{b \cdot v}{v \cdot v} v$

Proof: Let $b' = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} u_2 + \dots + \frac{b \cdot u_n}{u_n \cdot u_n} u_n$

We need $b - b' \in V^\perp$, i.e. $(b - b') \cdot u_i = 0$ for all i .

$$(b - b') \cdot u_1 = b \cdot u_1$$

$$- \left[\frac{b \cdot u_1}{u_1 \cdot u_1} \cancel{u_1 \cdot u_1} + \frac{b \cdot u_2}{u_2 \cdot u_2} \cancel{u_2 \cdot u_1} + \dots + \frac{b \cdot u_n}{u_n \cdot u_n} \cancel{u_n \cdot u_1} \right]$$

$$= b \cdot u_1 - b \cdot u_1 = 0$$

Do the same for $u_2, u_3, \dots$



Eg: Find the projection of $b = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix}$ onto
 $V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \right\}$

These vectors are **orthogonal**, so

$$b_v = \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{\begin{pmatrix} 1 \\ 2 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$= \frac{8}{4} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \frac{-2}{4} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 3/2 \\ 5/2 \\ 5/2 \end{pmatrix}$$

Projection Matrix: Outer Product Form

Let $\{u_1, \dots, u_n\}$ be an orthogonal set and let $V = \text{Span}\{u_1, \dots, u_n\}$. Then

$$P_V = \frac{u_1 u_1^T}{u_1 \cdot u_1} + \frac{u_2 u_2^T}{u_2 \cdot u_2} + \dots + \frac{u_n u_n^T}{u_n \cdot u_n}$$

NB: $n=1 \leadsto$ get projection onto a line $P_V = \frac{v v^T}{v \cdot v}$

NB: **outer product forms** of matrices will be a key part of the SVD.

Proof: $\left(\frac{u_1 u_1^T}{u_1 \cdot u_1} + \frac{u_2 u_2^T}{u_2 \cdot u_2} + \dots + \frac{u_n u_n^T}{u_n \cdot u_n} \right) b$

This is the defining property of P_V

$$= \frac{u_1}{u_1 \cdot u_1} (u_1^T b) + \dots + \frac{u_n}{u_n \cdot u_n} (u_n^T b)$$

$$= \frac{u_1 \cdot b}{u_1 \cdot u_1} u_1 + \dots + \frac{u_n \cdot b}{u_n \cdot u_n} u_n = b_V = P_V b \quad \checkmark$$

Eg: Find P_V for $V = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right\}$

$$P_V = \frac{1}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} + \frac{1}{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Now we consider **orthonormal** vectors.

Facts:

Let $\{v_1, \dots, v_n\}$ be an **orthonormal** set and let $Q = \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$.

(1) $Q^T Q = I_n$

(2) $(Qx) \cdot (Qy) = x \cdot y$ for all $x, y \in \mathbb{R}^n$

(3) $\|Qx\| = \|x\|$ for all $x \in \mathbb{R}^n$

(4) Let $V = \text{Span}\{v_1, \dots, v_n\} = \text{Col}(Q)$. Then

$$P_V = QQ^T$$

$$x \cdot y = \|x\| \cdot \|y\| \cdot \cos \theta$$

NB: (2) says $(Q \cdot)$ does not change **angles**.

(3) says $(Q \cdot)$ does not change **lengths**.

Proofs: (1) cf. p. 1 ✓

$$(2) (Qx) \cdot (Qy) = (Qx)^T Qy = x^T Q^T Qy = x^T I_n y = x \cdot y \quad \checkmark$$

$$(3) \|Qx\| = \sqrt{(Qx) \cdot (Qx)} \stackrel{(2)}{=} \sqrt{x \cdot x} = \|x\| \quad \checkmark$$

$$(4) P_V = Q(Q^T Q)^{-1} Q^T = Q(I_n)^{-1} Q^T = QQ^T \quad \checkmark$$

Eg: Find P_V for $V = \text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$

This has an orthonormal basis $\frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$$Q = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$P_V = Q Q^T = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

The projection formula is easier with no denominators:

Projection formula for an Orthonormal Basis:

Let $\{v_1, \dots, v_n\}$ be an orthonormal set and let $V = \text{Span}\{v_1, \dots, v_n\}$. For any vector b ,

$$b_V = (b \cdot v_1)v_1 + (b \cdot v_2)v_2 + \dots + (b \cdot v_n)v_n$$

Moreover,

$$P_V = v_1 v_1^T + v_2 v_2^T + \dots + v_n v_n^T$$

Def: A square matrix with orthonormal columns is called orthogonal.

↳ Note the strange terminology!

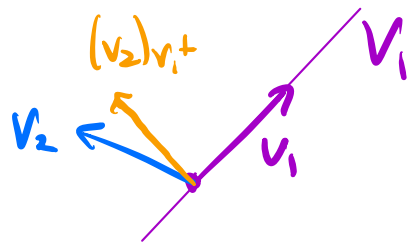
Q: Why is $P_V = QQ^T = I_n$ in this case?

Gram-Schmidt

Given that projections are easier to compute in an orthogonal basis, how do we produce one?

Idea: Start with any basis $\{v_1, \dots, v_n\}$

- make $v_2 \perp v_1$ by replacing with $(v_2)v_1^\perp$ for $V_1 = \text{Span}\{v_1\}$



- make $v_3 \perp v_1, v_2$ by replacing with $(v_3)v_2^\perp$ for $V_2 = \text{Span}\{v_1, v_2\}$
- etc

This "straightens out" the basis vectors one at a time.

NB: $(v_3)v_2^\perp$ is easy to compute w/ projection formula!

Procedure (Gram-Schmidt):

Let $\{v_1, \dots, v_n\}$ be a basis for a subspace V .

$$(1) u_1 := v_1$$

$$(2) u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = (v_2)_{v_1}^\perp \quad V_1 = \text{Span}\{v_1\}$$

$$(3) u_3 = v_3 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 = (v_3)_{V_2}^\perp \quad V_2 = \text{Span}\{v_1, v_2\}$$

$\vdots$

$$(n) u_n = v_n - \frac{u_1 \cdot v_n}{u_1 \cdot u_1} u_1 - \frac{u_2 \cdot v_n}{u_2 \cdot u_2} u_2 - \dots - \frac{u_{n-1} \cdot v_n}{u_{n-1} \cdot u_{n-1}} u_{n-1}$$

Then $\{u_1, \dots, u_n\}$ is an **orthogonal** basis for V , and

$$\text{Span}\{u_1, \dots, u_i\} = \text{Span}\{v_1, \dots, v_i\} \quad \text{for } 1 \leq i \leq n$$

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \quad v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$\begin{aligned} u_3 &= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \end{aligned}$$

output: $u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad u_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

check: $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0 \quad \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = 0$ ✓

Q: What if $\{v_1, \dots, v_n\}$ is linearly dependent?

Then eventually $v_i \in \text{Span}\{v_1, \dots, v_{i-1}\} = \text{Span}\{u_1, \dots, u_{i-1}\}$

so $v_i \in V_{i-1} = \text{Span}\{u_1, \dots, u_{i-1}\} \Rightarrow u_i = (v_i)_{V_{i-1}^\perp} = 0$

This is ok! Just discard v_i & continue.

QR Decomposition

This "keeps track" of the Gram-Schmidt procedure in the same way that LU keeps track of row operations.

Start with a basis $\{v_1, \dots, v_n\}$ of a subspace & run Gram-Schmidt. Then

Solve for v_i 's in terms of u_i 's:

$$v_1 = u_1$$

$$v_2 = \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 + u_2$$

$$v_3 = \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 + u_3$$

$$v_4 = \frac{v_4 \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{v_4 \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{v_4 \cdot u_3}{u_3 \cdot u_3} u_3 + u_4$$

Matrix Form:

$$\begin{pmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{pmatrix} \begin{pmatrix} 1 & \frac{v_2 \cdot u_1}{u_1 \cdot u_1} & \frac{v_3 \cdot u_1}{u_1 \cdot u_1} & \frac{v_4 \cdot u_1}{u_1 \cdot u_1} \\ 0 & 1 & \frac{v_3 \cdot u_2}{u_2 \cdot u_2} & \frac{v_4 \cdot u_2}{u_2 \cdot u_2} \\ 0 & 0 & 1 & \frac{v_4 \cdot u_3}{u_3 \cdot u_3} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

QR Decomposition:

Let A be an $m \times n$ matrix with full column rank.
Then

$$A = QR$$

where

- Q is an $m \times n$ matrix whose columns form an **orthonormal** basis of $\text{Col}(A)$
- R is **upper- Δ** $n \times n$ with nonzero diagonal entries.

To compute Q & R : let $\{v_1, \dots, v_n\}$ be the columns of A . Run Gram-Schmidt on $\{u_1, \dots, u_n\}$. Then

Q cancels R

$$\begin{pmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} & \frac{u_3}{\|u_3\|} & \frac{u_4}{\|u_4\|} \end{pmatrix} \begin{pmatrix} \|u_1\| & \frac{v_2 \cdot u_1}{u_1 \cdot u_1} \|u_1\| & \frac{v_3 \cdot u_1}{u_1 \cdot u_1} \|u_1\| & \frac{v_4 \cdot u_1}{u_1 \cdot u_1} \|u_1\| \\ 0 & \|u_1\| & \frac{v_3 \cdot u_2}{u_1 \cdot u_2} \|u_2\| & \frac{v_4 \cdot u_1}{u_2 \cdot u_2} \|u_2\| \\ 0 & 0 & \|u_3\| & \frac{v_4 \cdot u_3}{u_3 \cdot u_3} \|u_3\| \\ 0 & 0 & 0 & \|u_4\| \end{pmatrix}$$

Analogy to LU decomposition:

$$A = LU$$

↑
↑
 steps to get to echelon form echelon form

$$A = QR$$

↑
↑
 o.n. basis steps to get to o.n. basis

Eg: $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$ $v_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix}$

$u_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $\times \sqrt{2} = \|u_1\|$

$u_2 = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$u_3 = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \frac{6}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} - \begin{pmatrix} 3 \\ -3 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix}$ $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 3/\sqrt{2} \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$

$\times \sqrt{2} = \|u_1\|$
 $\times \sqrt{2} = \|u_2\|$
 $\times \sqrt{6} = \|u_3\|$

Application: makes least- $\square$ faster!

Given $A=QR$, to solve $Ax=b$ by least- $\square$:

$$A^T A \hat{x} = (QR)^T (QR) \hat{x} = R^T Q^T Q R \hat{x} = R^T I_n R \hat{x} = R^T R \hat{x}$$

$$A^T b = QR^T b = R^T Q^T b$$

NB: R is invertible: it is upper- Δ with nonzero diagonal entries.

Solve $R^T R \hat{x} = R^T Q^T b : (R^T)^{-1} \cdot (\cdot)$

$$A^T A \hat{x} = A^T b \iff R \hat{x} = Q^T b$$

R is upper- Δ : solve with back substitution!

NB: Can compute QR in $\sim \frac{10}{3} n^3$ flops for $n \times n$.
 (not with this algorithm) Then need $O(n^2)$ flops to do
 least- $\square$ on $Ax=b$. (Multiply by Q^T &
 forward-substitute.) Much faster than $O(n^3)$!

Eg: Find the least squares soln of $Ax=b$ for
 $A = \begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ using $A=QR$

for $Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \end{pmatrix}$ $R = \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{6} \end{pmatrix}$

$$Q^T b = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 4/\sqrt{6} \end{pmatrix}$$

$$R \hat{x} = Q^T b \rightsquigarrow \left(\begin{array}{cc|c} \sqrt{2} & \sqrt{2} & 0 \\ 0 & \sqrt{6} & 4/\sqrt{6} \end{array} \right) \rightsquigarrow \begin{array}{l} x_1 \sqrt{2} + x_2 \sqrt{2} = 0 \\ x_2 \sqrt{6} = \frac{4}{\sqrt{6}} \end{array}$$

$$\Rightarrow x_2 = \frac{4}{6} = \frac{2}{3}, \quad x_1 \sqrt{2} + \frac{2}{3} \sqrt{2} = 0$$

$$\Rightarrow x_1 = -\frac{2}{3} \Rightarrow \hat{x} = \begin{pmatrix} 2/3 \\ -2/3 \end{pmatrix}$$