Complex Eigenvalues

Some matrices have no (real) eigenvalues. But every matrix has a complex eigenvalue: any polynomial $p(\lambda)$ has a complex zero.

Eg: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (ccw rotation by 90°)

$$p(\lambda) = \lambda^2 + 1 = (\lambda + i)(\lambda - i)$$

Diagonalization still works great even if the eigenvalues are not real.

→ Still can solve difference equations & ODEs

→ Still get real-number answers.

So we can apply diagonalization techniques to more matrices if we allow complex eigenvalues.

Fact: The complex eigenvalues & eigenvectors of a real matrix come in complex conjugate pairs:

$$Av = \lambda v \iff A\bar{v} = \bar{\lambda}\bar{v}$$

here $v = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \rightsquigarrow \bar{v} = \begin{pmatrix} \bar{z}_1 \\ \vdots \\ \bar{z}_n \end{pmatrix}$

Eg: Solve the difference equation

$$v_{k+1} = A v_k \quad A = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix} \quad v_0 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

No complex #s in the statement!

(1) Diagonalize:

$$p(\lambda) = \lambda^2 + 3\lambda + 3 \Rightarrow \lambda = \tfrac{1}{2}\left(-3 \pm \sqrt{9-12}\right)$$

$$\rightarrow \lambda = \tfrac{1}{2}(-3 + i\sqrt{3}), \quad \bar{\lambda} = \tfrac{1}{2}(-3 - i\sqrt{3})$$

Find eigenvectors using the 2×2 trick

$$w = \begin{pmatrix} -b \\ a-\lambda \end{pmatrix} = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} \qquad \bar{w} = \begin{pmatrix} 1 \\ -\bar{\lambda} \end{pmatrix}$$

eigenvector for λ eigenvector for $\bar{\lambda}$

Check: $$Aw = \begin{pmatrix} 0 & -1 \\ 3 & -3 \end{pmatrix}\begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$$

Wait, is this equal to λw?

$$\lambda w = \lambda \begin{pmatrix} 1 \\ -\lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -\lambda^2 \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} \lambda \\ 3+3\lambda \end{pmatrix}$$

Yes: $-\lambda^2 = 3 + 3\lambda$ because

$$\lambda^2 + 3\lambda + 3 = p(\lambda) = 0 \ \checkmark$$

So $\{w, \bar{w}\}$ is an eigenbasis

(different eigenvalues $\Rightarrow$ LI)

(2) Expand the initial state in our eigenbasis:

We need to solve $\begin{pmatrix}2\\3\end{pmatrix} = v_0 = x_1 w + x_2 \bar{w}$.

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 1 & 2 \\ -\lambda & -\bar{\lambda} & 3 \end{array}\right) \xrightarrow{R_2 += \lambda R_1} \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & \lambda - \bar{\lambda} & 3+2\lambda \end{array}\right)$$

$3+2\lambda = i\sqrt{3}$

$\lambda - \bar{\lambda} = 2i\,\mathrm{Im}(\lambda) = i\sqrt{3}$

$$\rightsquigarrow \begin{cases} x_1 + x_2 = 2 \\ (i\sqrt{3})x_2 = i\sqrt{3} \end{cases} \rightsquigarrow x_2 = 1 \longrightarrow x_1 = 1$$

(back-substitution)

So $\boxed{v_0 = w_1 + w_2}$

Solution: $\boxed{A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w}}$

So far it's exactly the same as for real eigenvalues!

... but we wanted a solution involving only real #s.

Thankfully, $\lambda^k w$ and $\bar{\lambda}^k \bar{w}$ are complex conjugates, so

$$A^k v_0 = \lambda^k w + \bar{\lambda}^k \bar{w} = 2\mathrm{Re}[\lambda^k w]$$

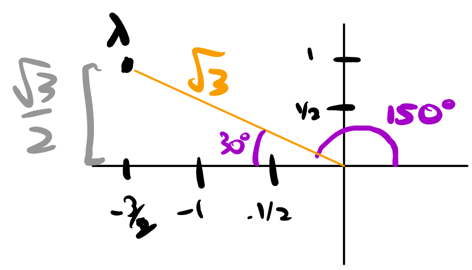
$$= 2\mathrm{Re}\left[\lambda^k \begin{pmatrix}1\\-\lambda\end{pmatrix}\right] = 2\mathrm{Re}\begin{pmatrix}\lambda^k\\-\lambda^{k+1}\end{pmatrix}$$

Recall: Multiplication of complex numbers is much easier in polar form.

$$\lambda = \tfrac{1}{2}(-3+i\sqrt{3}) = re^{i\theta}$$

$$r = \tfrac{1}{2}\sqrt{9+3} = \tfrac{1}{2}\sqrt{4\cdot 3} = \sqrt{3}$$

$$\theta = 150^\circ = 5\pi/6$$



Euler's Formula

So $$\lambda^k = r^k e^{ik\cdot\frac{5\pi}{6}} = (\sqrt{3})^k\left(\cos\frac{5k\pi}{6} + i\sin\frac{5k\pi}{6}\right)$$

$$\Rightarrow \operatorname{Re}(\lambda^k) = (\sqrt{3})^k \cos\frac{5k\pi}{6}$$

$$\Rightarrow \boxed{v_k = 2\begin{pmatrix} \sqrt{3}^k \cos(5k\pi/6) \\ -\sqrt{3}^{k+1}\cos(5(k+1)\pi/6)\end{pmatrix}}$$ [demo]

The answer involves only real numbers (and cosines-weird!) but we needed complex numbers to get it!

Difference Equations with Complex Eigenvalues:

To solve $v_{k+1} = Av_k$:

(1-2) Diagonalize A and expand v_0 in an eigenbasis, as before. Complex numbers are OK.

→ Remember $Av = \lambda v \Leftrightarrow A\bar{v} = \bar{\lambda}\bar{v}$

(3) Group complex conjugate terms:

$$\lambda^k x w + \bar{\lambda}^k \bar{x}\bar{w} = 2\operatorname{Re}(\lambda^k x w)$$

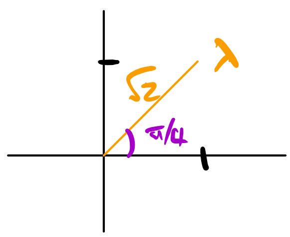
(4) Write λ in polar form:

$$\lambda = re^{i\theta} \Rightarrow \lambda^k = r^k e^{ik\theta} = r^k(\cos k\theta + i\sin k\theta)$$

Multiply this by x and the coordinates of w and take the real part

we get an answer with sines & cosines (but no i's).

Eg (for 4): $\lambda = 1+i \quad x = 3-2i \quad w = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$



$$\lambda = \sqrt{2}e^{i\pi/4} \Rightarrow \lambda^k = (\sqrt{2})^k e^{ik\pi/4}$$
$$= 2^{k/2}\left(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}\right)$$

$$\Rightarrow \lambda^k x w = 2^{k/2}\left(\cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}\right)(3-2i)\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$
$$= 2^{k/2}\left[3\cos\frac{k\pi}{4} + 2\sin\frac{k\pi}{4} + i\left(3\sin\frac{k\pi}{4} - 2\cos\frac{k\pi}{4}\right)\right]\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$
$$= 2^{k/2}\begin{pmatrix} 3\cos\frac{k\pi}{4} + 2\sin\frac{k\pi}{4} + i\left(3\sin\frac{k\pi}{4} - 2\cos\frac{k\pi}{4}\right) \\ -6\sin\frac{k\pi}{4} + 4\cos\frac{k\pi}{4} + i\left(6\cos\frac{k\pi}{4} + 4\sin\frac{k\pi}{4}\right) \end{pmatrix}$$

$$\Rightarrow 2\mathrm{Re}[\lambda^k x w] = 2\cdot 2^{k/2}\begin{pmatrix} 3\cos\frac{k\pi}{4} + 2\sin\frac{k\pi}{4} \\ -6\sin\frac{k\pi}{4} + 4\cos\frac{k\pi}{4} \end{pmatrix}$$

Algebraic & Geometric Multiplicity

Last we will discuss a criterion for diagonalizability.

(We like diagonalizable matrices because we can solve difference equations)

Recall: If λ is a root of a polynomial $p(x)$, its multiplicity m is the largest power of $(x-\lambda)$ dividing p:

$$p(x) = (x-\lambda)^m \cdot (\text{other factors})$$

Eg: $p(\lambda) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)^1$

$\lambda = 2$ has multiplicity 2; $\lambda = -1$ has multiplicity 1

Def: Let A be an $n \times n$ matrix with eigenvalue λ.

(1) The algebraic multiplicity (AM) of λ is its multiplicity as a root of the characteristic polynomial $p(\lambda)$.

(2) The geometric multiplicity (GM) of λ is the dimension of the λ-eigenspace:

$$\begin{aligned} GM(\lambda) &= \dim \operatorname{Nul}(A - \lambda I_n) \\ &= \#\text{free variables in } A - \lambda I_n. \\ &= \#\text{linearly independent } \lambda\text{-eigenvectors} \end{aligned}$$

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ $p(\lambda) = -(\lambda-2)^{2}(\lambda-1)^{1}$

So the eigenvalues are 1 & 2.

- $\lambda = 1$: $AM = 1$.

  $\mathrm{Nul}(A - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ $\quad AM \geq GM$ ✓

  $\rightarrow$ this is a line: $GM = 1$

- $\lambda = 2$: $AM = 2$

  $\mathrm{Nul}(A - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}\right\}$ $\quad AM \geq GM$ ✓

  $\rightarrow$ this is a line: $GM = 1$

This matrix is not diagonalizable:

only two linearly independent eigenvectors.

[demo]

Eg: $B=\begin{pmatrix} -4 & 3 & 2 \\ -6 & 5 & 2 \\ -6 & 3 & 4 \end{pmatrix}$ $\quad p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$

So the eigenvalues are 1 & 2.

- $\lambda = 1$: AM $= 1$

  $\mathrm{Nul}(B - 1I_3) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$ $\quad$ AM=GM ✓

  → this is a line: GM $= 1$

- $\lambda = 2$: AM $= 2$

  $\mathrm{Nul}(B - 2I_3) = \mathrm{Span}\left\{\begin{pmatrix}3\\4\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$ $\quad$ AM=GM ✓

  → this is a plane: GM $= 2$

This matrix is diagonalizable: an eigenbasis is

$$\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}3\\4\\3\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$$

[demo]

Both matrices have only 2 eigenvalues.

The difference is that B had AM=GM=2 LI 2 eigenvectors and A had one.

Thm ($AM \geq GM$): For any eigenvalue λ of A,

(algebraic multiplicity of λ)

$\geq$ (geometric multiplicity of λ) ≥ 1

For a proof, see the supplement.

NB: $GM \geq 1$ just says every eigenvalue has an eigenvector — the eigenspace can't be $\{0\}$, so its dimension is ≥ 1.

Upshot: if $p(\lambda) = -(\lambda-2)^2(\lambda-1)^1$ then

- the 1-eigenspace is necessarily a line:

  $$AM = 1 \geq GM \geq 1$$

- the 2-eigenspace is a line or a plane:

  $$AM = 2 \geq GM \geq 1$$

- the matrix is diagonalizable $\iff GM(2) = 2$: then you have $1+2=3$ LI eigenvectors.

Thm (AM/GM Criterion for Diagonalizability):

Let A be an $n \times n$ matrix.

- A is diagonalizable over the complex numbers $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ
- A is diagonalizable over the real numbers $\iff AM(\lambda) = GM(\lambda)$ for every eigenvalue λ and A has no complex eigenvalues.

Eg: $A = \begin{pmatrix} -7 & 3 & 5 \\ -10 & 5 & 6 \\ -9 & 3 & 7 \end{pmatrix}$ is not diagonalizable because $AM(2) = 2 \neq 1 = GM(2)$

Corollary: If A has n different eigenvalues then A is diagonalizable.

Proof: If A has n different eigenvalues then

$$n = AM(\lambda_1) + \cdots + AM(\lambda_n) \Rightarrow AM(\lambda_i) = 1$$

$$1 = AM(\lambda_i) \geq GM(\lambda_i) \geq 1 \Rightarrow AM(\lambda_i) = GM(\lambda_i) = 1$$

Eg: A 2×2 real matrix with a complex eigenvalue λ is diagonalizable (over $\mathbb{C}$): it has 2 eigenvalues λ and $\bar{\lambda}$.

Proof of the Theorem:

First note that

$$p(\lambda)=(-1)^n(\lambda-\lambda_1)^{m_1}\cdots(\lambda-\lambda_r)^{m_r}$$

factors into linear factors (over $\mathbb{C}$), where $m_i = AM(\lambda_i)$. Hence

$$\boxed{AM(\lambda_1)+\cdots+AM(\lambda_r)=n \quad \text{(sum of the AM's is } n\text{)}}$$

If A is diagonalizable then it has n LI eigenvectors, so

$$\begin{aligned} n = GM(\lambda_1) &+ \cdots + GM(\lambda_n) \\ \wedge\!\!\mid \quad & \quad\quad \wedge\!\!\mid \\ AM(\lambda_1) &+ \cdots + AM(\lambda_n) = n \end{aligned}$$

This forces $AM(\lambda_i)=GM(\lambda_i)$. Conversely, if each $AM(\lambda_i)=GM(\lambda_i)$ then

$$n = GM(\lambda_1)+\cdots+GM(\lambda_n),$$

so when you combine eigenspace bases you get n LI eigenvectors.