Systems of ODEs

Toy Example: Here is an extremely simplistic model of disease spread:

$H(t)$ = # healthy people at time t (in years)
$I(t)$ = # infected people at time t
$D(t)$ = # dead people at time t

Assumptions:

(1) Healthy people are infected at a rate of
$0.3 \times$ #healthy people
(2) Infected people recover at a rate of
$0.9 \times$ #infected people
(3) Infected people die at a rate of
$0.1 \times$ #infected people

In equations:

(1) $$\frac{dH}{dt} = \underbrace{-0.3H}_{\text{infected}} + \underbrace{0.9I}_{\text{recovered}}$$

(2) $$\frac{dI}{dt} = \underbrace{0.3H}_{\text{infected}} - \underbrace{0.9I}_{\text{recovered}} - \underbrace{0.1I}_{\text{dead}}$$

(3) $$\frac{dD}{dt} = \underbrace{0.1I}_{\text{dead}}$$

Matrix Form: let $u(t) = (H(t), I(t), D(t))$.

$$\frac{du(t)}{dt} = u'(t) = \begin{bmatrix} -0.3 & 0.9 & 0 \\ 0.3 & -0.9-0.1 & 0 \\ 0 & 0.1 & 0 \end{bmatrix} u(t)$$

Def: A system of linear ordinary differential equations (ODEs) is a system of equations in unknown functions $u_1(t), \dots, u_n(t)$ equating the derivatives u_i' with a linear combination of the u_i:

$$\begin{aligned} u_1'(t) &= a_{11}u_1(t) + \cdots + a_{1n}u_n(t) \\ &\vdots \\ u_n'(t) &= a_{n1}u_1(t) + \cdots + a_{nn}u_n(t) \end{aligned}$$

Matrix form: writing $u(t) = (u_1(t), \dots, u_n(t))$ and $u'(t) = (u_1'(t), \dots, u_n'(t))$, a system of linear ODEs has the form

$$u'(t) = Au(t)$$

for an $n \times n$ matrix A

(with numbers in it, not functions of t).

If you also specify the initial value $u(0) = u_0$, this is called an initial value problem.

some vector

How to solve a system of linear ODEs?

Diagonalize A!

Eg: Suppose u_0 is an eigenvector of A: $Au_0 = \lambda u_0$.
Then the solution of the initial value problem

$$u' = Au \qquad u(0) = u_0 \quad \text{is} \quad u(t) = e^{\lambda t} u_0 :$$

does not depend on t

$$u'(t) = \frac{d}{dt} e^{\lambda t} u_0 = \lambda e^{\lambda t} u_0$$

$$Au(t) = Ae^{\lambda t} u_0 = e^{\lambda t} A u_0 = \lambda e^{\lambda t} u_0$$

scalar

$$u(0) = e^{\lambda 0} u_0 = u_0 \quad \checkmark$$

In general, we expand u_0 in an eigenbasis, as for difference equations:

$$u_0 = x_1 w_1 + \cdots + x_n w_n \qquad A w_i = \lambda_i w_i$$

$$\rightsquigarrow u(t) = e^{\lambda_1 t} x_1 w_1 + \cdots + e^{\lambda_n t} x_n w_n$$

is the solution of $u' = Au$, $u(0) = u_0$.

Check:

$$u'(t) = \lambda_1 e^{\lambda_1 t} x_1 w_1 + \cdots + \lambda_n e^{\lambda_n t} x_n w_n$$

$$Au(t) = e^{\lambda_1 t} x_1 A w_1 + \cdots + e^{\lambda_n t} x_n A w_n$$

$$= \lambda_1 e^{\lambda_1 t} x_1 w_1 + \cdots + \lambda_n e^{\lambda_n t} x_n w_n$$

$$u(0) = e^{\lambda_1 0} x_1 w_1 + \cdots + e^{\lambda_n 0} x_n w_n = u_0 \quad \checkmark$$

Eg: In our infectious disease model, suppose

$u_0 = (1000, 1, 0)$ (1000 healthy people, 1 infected, 0 dead)

Eigenvalues of $A = \begin{pmatrix} -0.3 & .9 & 0 \\ 0.3 & -1 & 0 \\ 0 & .1 & 0 \end{pmatrix}$ are

$\lambda_1 \approx -.0235$

$\lambda_2 \approx -1.28$

$\lambda_3 = 0$

Eigenvectors are

$w_1 \approx \begin{pmatrix} 11.77 \\ -12.77 \\ 1 \end{pmatrix}$ $w_2 \approx \begin{pmatrix} -.765 \\ -.235 \\ 1 \end{pmatrix}$ $w_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

Solve $u_0 = x_1 w_1 + x_2 w_2 + x_3 w_3$:

$$u_0 = \begin{pmatrix} 1000 \\ 1 \\ 0 \end{pmatrix} \approx 18.70\, w_1 - 1019.70\, w_2 + 1001\, w_3$$

Solution is:

$$u(t) = e^{-.0235t} \cdot 18.70 w_1 - e^{-1.28t} \cdot 1019.70 w_2 + 1001 w_3$$

$$\Rightarrow \begin{aligned} H(t) &= 220 e^{-.0235t} + 780 e^{-1.28t} \\ I(t) &= -238 e^{-.0235t} + 239 e^{-1.28t} \\ D(t) &= 18.7 e^{-.0235t} - 1019.7 e^{-1.28t} + 1001 \end{aligned}$$

Looks like the human race is doomed...

Procedure for solving a linear system of ODEs using diagonalization:

To solve $u' = Au$, $u(0) = u_0$ when A is diagonalizable:

(1) Diagonalize A: get an eigenbasis $\{w_1, \dots, w_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$.

(2) Expand u_0 in the eigenbasis: solve $u_0 = x_1 w_1 + \dots + x_n w_n$

Solution:

$$u(t) = e^{\lambda_1 t} x_1 w_1 + \dots + e^{\lambda_n t} x_n w_n$$

Compare to:

Procedure for solving a Difference Equation using diagonalization:

To solve $v_{k+1} = A v_k$, v_0 fixed when A is diagonalizable:

(1) Diagonalize A: get an eigenbasis $\{w_1, \dots, w_n\}$ with eigenvalues $\lambda_1, \dots, \lambda_n$.

(2) Expand v_0 in the eigenbasis: solve $v_0 = x_1 w_1 + \dots + x_n w_n$

Solution:

$$v_k = \lambda_1^k x_1 w_1 + \dots + \lambda_n^k x_n w_n$$

This works fine with complex eigenvalues. As with difference equations, you can write the solution with real numbers using trig functions.

Eg: $u_1'(t) = u_2, \quad u_2'(t) = -4u_1,$

$u_1(0) = 2 \qquad u_2(0) = 0$

$$\rightsquigarrow u' = Au \quad \text{for} \quad A = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \quad u(0) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

Eigenvalues are $\lambda = 2i, \quad \bar{\lambda} = -2i$

Eigenvectors are $w = \begin{pmatrix} 1 \\ 2i \end{pmatrix} \quad \bar{w} = \begin{pmatrix} 1 \\ -2i \end{pmatrix}$

Solve $\begin{pmatrix} 2 \\ 0 \end{pmatrix} = x_1 w + x_2 \bar{w} \rightsquigarrow x_1 = x_2 = 1$

Solution is

$(x_1 = x_2 = 1)$

$$u(t) = e^{\lambda t} w + e^{\bar{\lambda} t} \bar{w} = 2\mathrm{Re}\left[e^{\lambda t} w\right]$$

$$= 2\mathrm{Re}\left[e^{2it}\begin{pmatrix} 1 \\ 2i \end{pmatrix}\right] = 2\mathrm{Re}\left[(\cos(2t) + i\sin(2t))\begin{pmatrix} 1 \\ 2i \end{pmatrix}\right]$$

$$= 2\mathrm{Re}\begin{pmatrix} \cos(2t) + i\sin(2t) \\ -2\sin(2t) + 2i\cos(2t) \end{pmatrix} = \begin{pmatrix} 2\cos(2t) \\ -4\sin(2t) \end{pmatrix}$$

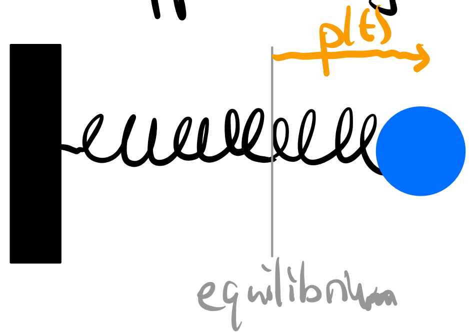
Check: $u_1' = (2\cos(2t))' = -4\sin(2t) = u_2$

$u_2' = (-4\sin(2t))' = -8\cos(2t) = -4u_1$

$u_1(0) = 2 \quad u_2(0) = 0$ ✓

This method can also be used to solve (linear) ODEs containing higher-order derivatives.

Eg: Hooke's Law says the force applied by a spring is proportional to the amount it is stretched or compressed:



$$F(t) = -k\, p(t) \qquad k>0$$

$F=ma$, $a = \text{acceleration} = p''$: replace k by k/m:

$$p''(t) = -k\,p(t)$$

Trick: Let $u_1 = p$, $u_2 = p'$. Then

$$u_1' = u_2 \qquad u_2' = -k u_1.$$

This is the system

$$u'(t) = \begin{pmatrix} 0 & 1 \\ -k & 0 \end{pmatrix} u(t).$$

We solved this before for $k=4$, $u(0) = (2,0)$:

$$p(t) = 2\cos(2t)$$
$$p'(t) = -4\sin(2t)$$

oscillation.

The Matrix Exponential

There are 2 features missing from the ODEs picture that we had for difference equations:

(1) Matrix form: $v_k = CD^kC^{-1}v_0$

(2) Existence of solutions:

it's obvious that $v_k = A^k v_0$ has a solution

– it was not obvious how to compute it.

Both can be filled in using the matrix exponential.

Recall: Using Taylor expansions, you can write

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots \qquad \text{(convergent sum)}$$

Def: Let A be an $n \times n$ matrix. The matrix exponential is the $n \times n$ matrix

$$e^A = I_n + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots \qquad \text{(convergent sum)}$$

Eg: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow A^2 = 0$, so

$$e^{At} = I_2 + At + 0 + \cdots = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Eg: $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \rightsquigarrow A^k = \begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix}$, so

$$e^{At} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_1 t & 0 \\ 0 & \lambda_2 t \end{pmatrix} + \begin{pmatrix} \frac{1}{2!}\lambda_1^2 t^2 & 0 \\ 0 & \frac{1}{2!}\lambda_2^2 t^2 \end{pmatrix} + \begin{pmatrix} \frac{1}{3!}\lambda_1^3 t^3 & 0 \\ 0 & \frac{1}{3!}\lambda_2^3 t^3 \end{pmatrix} + \cdots$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Why do we care about e^{At}?

Fact: $\boxed{\frac{d}{dt} e^{At} = A e^{At}}$

Consequence: $u(t) = e^{At} u_0$ solves the linear ODE

$$u'(t) = Au(t) \qquad u(0) = u_0$$

In particular, a solution exists.

The equations

$$u(t) = e^{At} u_0 \qquad \text{and} \qquad v_k = A^k v_0$$

are analogous: they both show a solution exists, but give you no way to compute it.

Eg: If $A = CDC^{-1}$ is diagonalizable then

$$e^{At} = Ce^{Dt}C^{-1} = C \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} C^{-1}.$$

This is computable!

The equations

$$e^{At} = Ce^{Dt}C^{-1} \quad \text{and} \quad A^k = CD^kC^{-1}$$

are also analogous: they are computable!

In fact, if you expand out

$$u(t) = Ce^{Dt}C^{-1}u_0$$

you exactly get the vector form

$$u(t) = e^{\lambda_1 t}x_1 w_1 + \cdots + e^{\lambda_n t}x_n w_n$$

where $(x_1, \ldots, x_n) = C^{-1}u_0$.

Difference Equation	Dictionary	Initial Value Problem
$V_{k+1} = AV_k$ V_0 fixed	Problem	$u'(t) = Au(t)$ $u(0)$ fixed
$V_k = A^k V_0$	Uncomputable Solution	$u(t) = e^{At}u(0)$
$V_k = \lambda_1^k x_1 w_1 + \cdots + \lambda_n^k x_n w_n$ for $V_0 = x_1 w_1 + \cdots + x_n w_n$	Computable Solution (when diagonalizable)	$u(t) = e^{\lambda_1 t}x_1 w_1 + \cdots + e^{\lambda_n t}x_n w_n$ for $u(0) = x_1 w_1 + \cdots + x_n w_n$
$A^k = CD^kC^{-1}$	Matrix Form	$e^{At} = Ce^{Dt}C^{-1}$