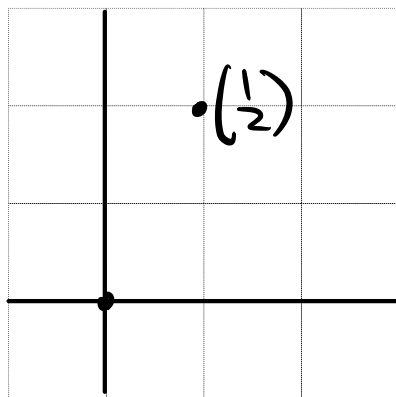


# Geometry of Vectors

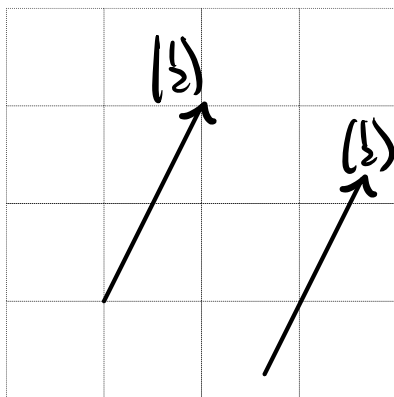
Recall: A **vector** in  $\mathbb{R}^n$  is a list of  $n$  numbers:  
$$v = (x_1, \dots, x_n) \in \mathbb{R}^n$$

We can draw a vector  
as a **point** in Euclidean space:  
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{x-coordinate} \\ \text{y-coordinate} \end{pmatrix}$$



We will often consider a vector as an **arrow**, or **displacement**: measures the **difference** between two points.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \text{x-displacement} \\ \text{y-displacement} \end{pmatrix}$$



**NB** the tail of the vector can be anywhere, but by default vectors start at 0

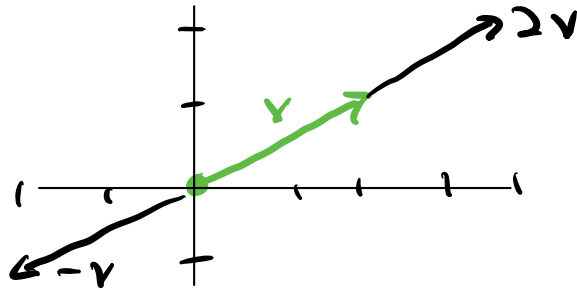
How do **algebraic operations** behave **geometrically**?  
We'll describe in terms of arrows.

## Scalar Multiplication:

- the **length** of  $cv$  is  $|c| \times$  the length of  $v$
- the **direction** of  $cv$  is
  - the same as  $v$  if  $c > 0$
  - the opposite from  $v$  if  $c < 0$

[demo]

Eg:  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$$2v = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

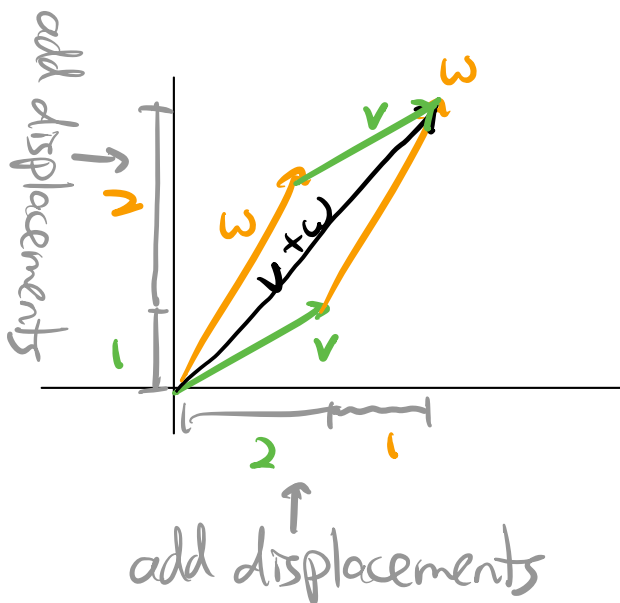
$$-v = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

## Vector Addition:

This just adds the **displacements**.

**Parallelogram Law:** to draw  $v+w$ , draw the **tail** of  $v$  at the **head** of  $w$  (or vice-versa); the head of  $v$  is at  $v+w$ .

Eg:  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$   
 $w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$   
 $v+w = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$



[demo]

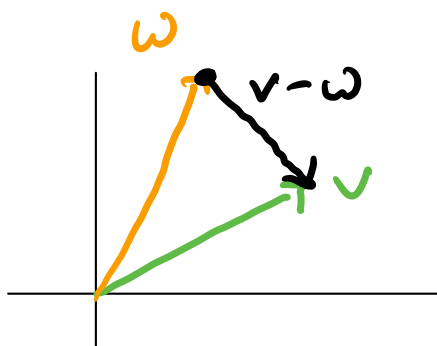
Vector Subtraction:  $w + (v - w) = v$

So  $v - w$  starts at the head of  $w$  & ends at the head of  $v$ .

Eg:  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$v - w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



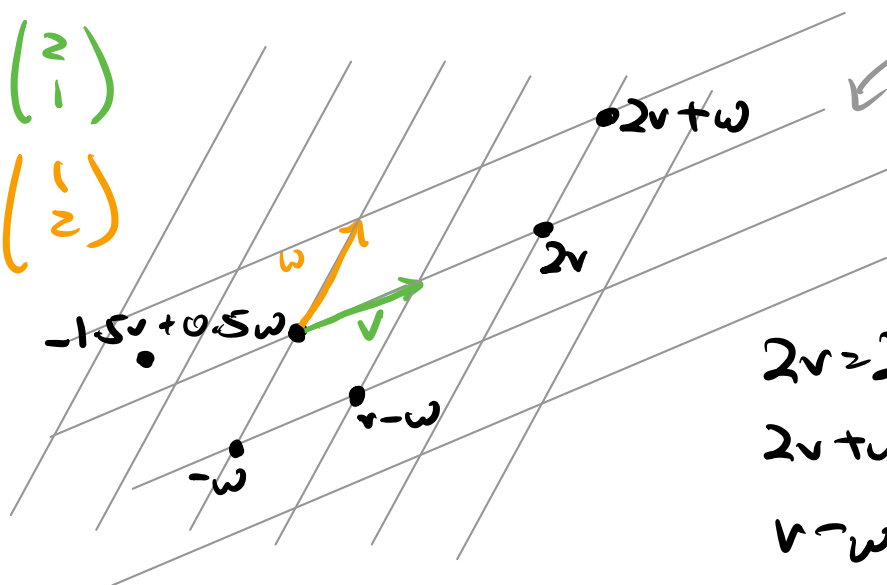
[demo]

Linear Combinations:

First scale, then add.

Eg:  $v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$



grid lines are in  $v$ - and  $w$ -directions

[demos]

$2v = 2v + 0w = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$

$2v + w = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$

$v - w = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

This is like giving directions: "To get to  $-1.5v + 0.5w$ , first go  $1.5 \times$  length of  $v$  in the opposite  $v$ -direction, then go  $0.5 \times$  length of  $w$  in the  $w$ -direction."

Spans look out for two subtle concepts below.

Recall: the notion of "all linear combinations of some set of vectors" came up twice last time:

- The solution set of  $Ax=b$  is  
(particular solution) + (all linear combinations of some vectors)
- $Ax=b$  is consistent if  
 $b \in$  (all linear combinations of the columns of  $A$ )

Def: The span of a list of vectors is the set of all linear combinations of those vectors:

$$\text{Span}\{v_1, v_2, \dots, v_n\} = \left\{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n : c_1, \dots, c_n \in \mathbb{R} \right\}$$

↑                      ↑                      ↑                      ↑  
"the set of"      "all things of this form"      "such that"      "these conditions hold"

This is set-builder notation ↑

Translation of the above:

(1) The solution set of  $Ax=b$  is  
(particular solution) +  $\text{Span}\{\text{some vectors}\}$

(2)  $Ax=b$  is consistent  $\Leftrightarrow b \in \text{Span}\{\text{columns of } A\}$



## Column Picture Criterion for Consistency (again)

$Ax=b$  is consistent (has at least one solution)



$b \in \text{Span}\{\text{columns of } A\}$

subtle  
concept  
#1

## What do spans look like?

It's the smallest "linear space" (line, plane, etc.) containing all your vectors & the origin.

Eg:  $\text{Span}\{v\} = \{cv : c \in \mathbb{R}\}$

→ If  $v \neq 0$  get the line thru  $0$  &  $v$

→  $\text{Span}\{0\} = \{c \cdot 0 : c \in \mathbb{R}\} = \{0\}$

= the set containing only  $0$

[demo]

Eg:  $\text{Span}\{v, w\} = \{cv + dw : c, d \in \mathbb{R}\}$

→ If  $v, w$  are not collinear, get the plane defined by  $0, v$ , and  $w$

→ If  $v, w$  are collinear and nonzero, get the line thru  $v, w$ , and  $0$ .

→ If  $v=w=0$  get  $\{0\}$

[demo]

Eg:  $\text{Span}\{u, v, w\} = \{bu + cv + dw : b, c, d \in \mathbb{R}\}$

→ If  $u, v, w$  are not coplanar, get **space**

→ If  $u, v, w$  are coplanar but not collinear, get the **plane** containing them.

→ If  $u, v, w$  are collinear & not all zero, get the **line** thru  $u, v, w$ , and  $0$ .

→ If  $u = v = w = 0$  get  $\{0\}$

[demo]

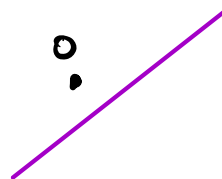
Eg:  $\text{Span}\{\overset{\text{empty set}}{\emptyset}\} = \{0\}$  (by convention)

**NB:** Every span contains the zero vector!

$$0 = 0 \cdot v_1 + 0 \cdot v_2 + \dots + 0 \cdot v_n$$

So eg. this **line** is **not a span**:

0



Eg:  $\{\emptyset\}$  is not a span! It does not contain  $0$ .

Q: Is  $\begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix}$  in  $\text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix}\right\}$ ?

In other words, does

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 3 \end{bmatrix} \quad \text{have a solution?}$$

Let's solve this **vector equation**:

$$\left[ \begin{array}{cc|c} 1 & -1 & 8 \\ 2 & -2 & 16 \\ 6 & -1 & 3 \end{array} \right] \xrightarrow{\text{RREF}} \left[ \begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array} \right] \rightsquigarrow \begin{array}{l} x_1 = -1 \\ x_2 = -9 \end{array}$$

Answer: yes

This example is just the " $\Leftarrow$ " of the statement:  
 $Ax=b$  is consistent  $\iff b \in \text{Span}\{\text{cols of } A\}$

Column Picture Criterion for Consistency:

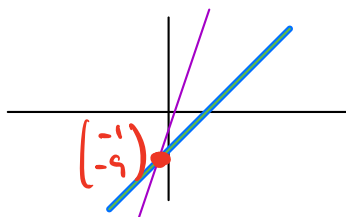
subtle  
concept  
#1

•  $\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} x = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$  is **consistent** because

$$\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix} \in \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\} \quad [\text{demo}]$$

row picture:

$$\begin{array}{l} x_1 - x_2 = 8 \\ 2x_1 - 2x_2 = 16 \\ 6x_1 - x_2 = 3 \end{array}$$

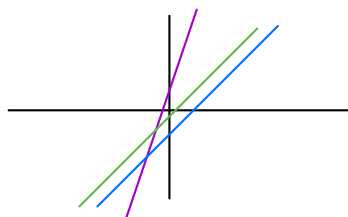


•  $\begin{pmatrix} 1 & -1 \\ 2 & -2 \\ 6 & -1 \end{pmatrix} x = \begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$  is **inconsistent** because

$$\begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix} \notin \text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} \right\} \quad [\text{demo}]$$

row picture:

$$\begin{array}{l} x_1 - x_2 = 7 \\ 2x_1 - 2x_2 = 1 \\ 6x_1 - x_2 = -1 \end{array}$$



# Homogeneous Equations

If the solution set of  $Ax=b$  is a span

$\Rightarrow 0$  is a solution (every span contains 0)

$$\Rightarrow A0=b \Rightarrow b=0$$

Let's study this case.

Def:  $Ax=b$  is called **homogeneous** if  $b=0$ .

Eg:  $x_1 + 2x_2 + 2x_3 + x_4 = 0$

$$2x_1 + 4x_2 + x_3 - x_4 = 0$$

NB: A homogeneous equation is **always consistent**  
since  $0$  is a solution:  $A \cdot 0 = 0$

Def: The **trivial solution** of a homogeneous equation  $Ax=0$  is the zero vector.

Eg: Let's solve the homogeneous system

$$x_1 + 2x_2 + 2x_3 + x_4 = 0$$

$$2x_1 + 4x_2 + x_3 - x_4 = 0 \rightsquigarrow \left[ \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \leftarrow 2R_1} \left[ \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 \end{array} \right]$$

$$\xrightarrow{R_2 \div -3} \left[ \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

$$\xrightarrow{R_1 \leftarrow 2R_2} \left[ \begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right]$$

$$\begin{aligned} \text{PF} \quad x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= -x_4 \\ x_4 &= x_4 \end{aligned}$$

$$\text{PVF} \quad x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Observations:

(1) The augmented column is **always zero**.

When solving homogeneous equations, you don't need to write the augmented column.

$$\left[ \begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array} \right] \rightsquigarrow \left[ \begin{array}{cccc} 1 & 2 & 2 & 1 \end{array} \right]$$

(2) The particular solution is the zero vector

(3) The solution set is

$$\text{Span} \left\{ \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

**Fact:** The PVF of a homogeneous system always has **particular solution = 0**. The solution set is the **span** of the other vectors you've produced.

# Inhomogeneous Equations

Def:  $Ax=b$  is called **inhomogeneous** if  $b \neq 0$ .

What's the difference from homogeneous equations?

**NB:** It can be inconsistent!

Let's solve the inhomogeneous & homogeneous versions:

Eg: **inhomogeneous**

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$\downarrow$  (augmented) matrix

$$\left[ \begin{array}{ccc|c} 2 & 1 & 12 & 1 \\ 1 & 2 & 9 & -1 \end{array} \right]$$

$\downarrow$  RREF

$$\left[ \begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 1 & 2 & -1 \end{array} \right]$$

$\downarrow$  PVF

$$x = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

$\downarrow$  Solution set

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix} \right\}$$

**homogeneous**

$$\begin{bmatrix} 2 & 1 & 12 \\ 1 & 2 & 9 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\downarrow$

$$\left[ \begin{array}{ccc|c} 2 & 1 & 12 & 0 \\ 1 & 2 & 9 & 0 \end{array} \right]$$

$\downarrow$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

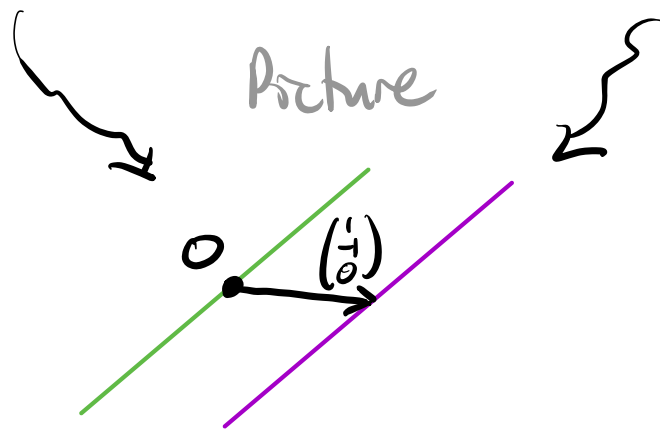
$\downarrow$

$$x = z \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix}$$

$\downarrow$

$$\text{Span} \left\{ \begin{pmatrix} -5 \\ -2 \\ 1 \end{pmatrix} \right\}$$

same



[demo]

The only difference is the particular solution!  
Otherwise they're parallel lines.

Facts:

subtle  
concept  
#2

- (1) The solution set of  $Ax=0$  is a span.
- (2) The solution set of  $Ax=b$  is not a span for  $b \neq 0$ : it is a translate of the solution set of  $Ax=0$  by a particular solution. (Or it is empty.)

$$\left( \begin{array}{l} \text{solutions} \\ \text{of } Ax=0 \end{array} \right) = (\text{zero}) + \text{Span} \left\{ \begin{array}{l} \text{vectors} \\ \text{from AVE} \end{array} \right\}$$

same vectors! ↕

$$\left( \begin{array}{l} \text{solutions} \\ \text{of } Ax=b \end{array} \right) = \left( \begin{array}{l} \text{particular} \\ \text{solution} \end{array} \right) + \text{Span} \left\{ \begin{array}{l} \text{vectors} \\ \text{from AVE} \end{array} \right\}$$

In fact, to get the solutions of  $Ax=b$  you can translate the solutions of  $Ax=0$  by **any** single solution of  $Ax=b$ .

→ Say  $p$  is some solution of  $Ax=b$ , so  $Ap=b$ .  
Then  $Ax=0 \iff Ap+Ax=b \iff A(p+x)=b$

vectors of the form  $p + (\text{soln of } Ax=0)$

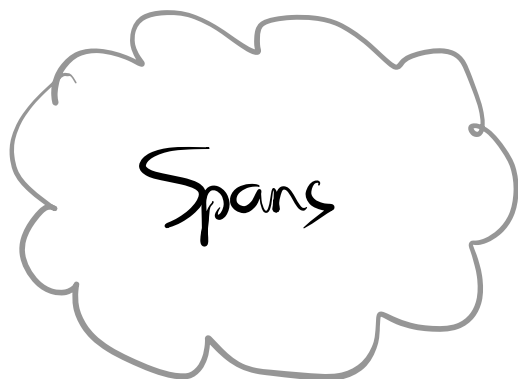
**NB:** Expressing a solution set as a (translate of a) span means writing it in **parametric form**:

$$x \in \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \text{Span} \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

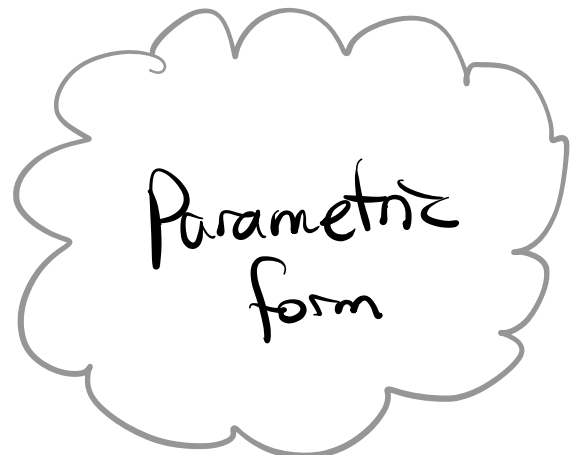
$$\iff x = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + d \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

↑ parameters

So think:



$\equiv$





# Row & Column Picture

We now know:

**Row Picture** (1) (All solutions of  $Ax=b$ )

subtle concept #2

$$= \left( \text{Some solution of } Ax=b \right) + \left( \text{All solutions of } Ax=0 \right)$$

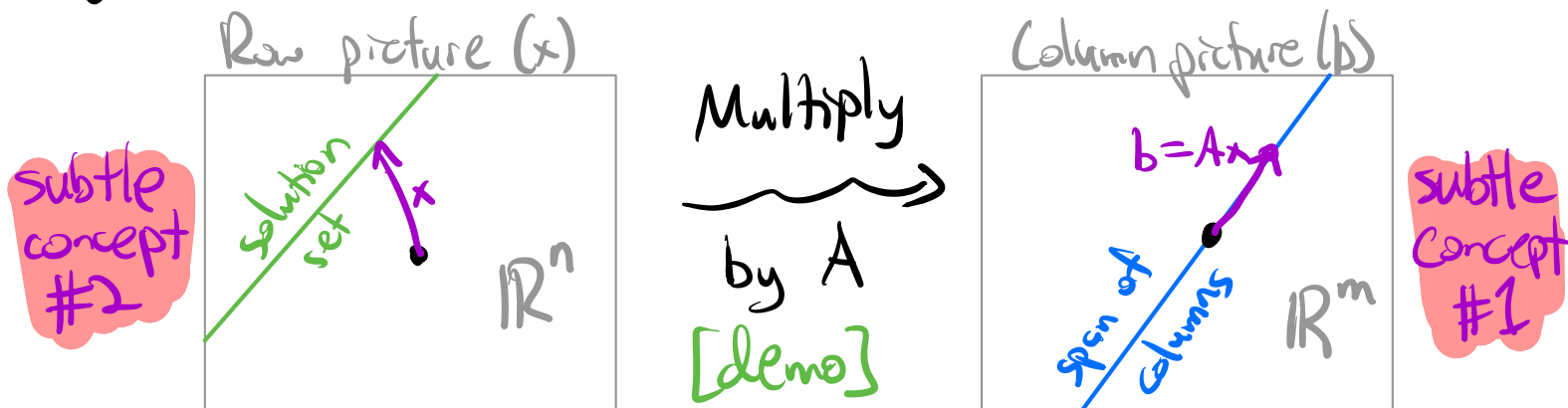
Span

or is empty. In particular, all nonempty solution sets are parallel and look the same.

**Column Picture** (2)  $Ax=b$  is consistent  $\iff b$  is in the span of the columns of  $A$ .

subtle concept #1

We can draw these both at the same time:



In this picture, we think of  $A$  as a function:

$x \in \mathbb{R}^n$  is the input (row picture)

$Ax \in \mathbb{R}^m$  is the output (column picture)

Solving  $Ax=b$  means finding all inputs with output  $= b$ .



The solution set lives in the... row picture!

The b-vectors live in the... column picture!

The columns all live in the... column picture!

That's how you keep them straight.