# Subspaces

(least squares)

**Orientation:** We're developing machinery to "almost solve" $Ax=b$

**Today:** give new names to everything we've been doing

So far, to every matrix $A$ we have associated two spans:

(1) the span of the columns / all $b$ such that $Ax=b$ is consistent

(2) the solution set of $Ax=0$

The first arises naturally as a span / it is already in parametric form. The second required work (elimination) to write as a span – it is a solution set, so it is in implicit form.

The notion of subspaces puts both on the same footing. This formalizes what we mean by "linear space containing 0".

**Fast-forward:**

same picture

$$\boxed{\text{Subspaces are spans} \quad \text{and} \quad \text{Spans are subspaces.}}$$

Why the new vocabulary word?

When you say "span" you have a spanning set of vectors in mind (parametric form). This is not the case for the solutions of $Ax=0$.

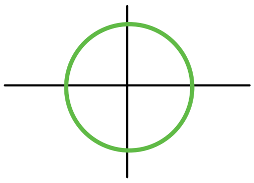
Subspaces allow us to discuss spans without computing a spanning set. Subspace = Span{ ???? }

They also give a criterion for a subset to be a span.

**Def:** A subset of $\mathbb{R}^n$ is any collection of points.

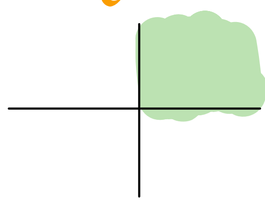
**Eg:**

(a)



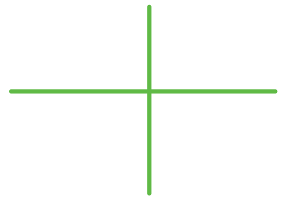
$\{(x,y) : x^2+y^2=1\}$

(b)



$\{(x,y) : x,y \geq 0\}$

(c)



$\{(x,y) : xy=0\}$

**Def:** A subspace is a subset $V$ of $\mathbb{R}^n$ satisfying:

(1) [closed under +] If $u,v \in V$ then $u+v \in V$

(2) [closed under scalar ×]

If $u \in V$ and $c \in \mathbb{R}$ then $cu \in V$

(3) [contains 0] $0 \in V$

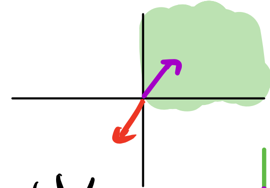
These conditions characterize linear spaces containing 0 among all subsets.

**NB:** If $V$ is a subspace and $v \in V$ then $0 = 0v$ is in $V$ by (2), so (3) just means $V$ is nonempty

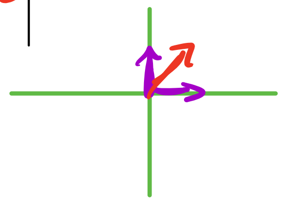
Eg: In the subsets above:

(a) fails (1), (2), (3)

(b) fails (2): $\begin{pmatrix}1\\1\end{pmatrix} \in V$ but $-1\cdot\begin{pmatrix}1\\1\end{pmatrix} \notin V$



(c) fails (1): $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \in V$ but $\begin{pmatrix}1\\1\end{pmatrix} \notin V$



Here are two "trivial" examples of subspaces:

Eg: $\{0\}$ is a subspace

(1) $0+0=0\in\{0\}$ ✓

(2) $c\cdot 0 = 0 \in \{0\}$ ✓

(3) $0\in\{0\}$ ✓

NB $\{0\} = \mathrm{Span}\{\}$: it is a span

Eg: $\mathbb{R}^n = \{$all vectors of size $n\}$ is a subspace

(1) The sum of two vectors is a vector. ✓

(2) A scalar times a vector is a vector. ✓

(3) $0$ is a vector. ✓

NB $\mathbb{R}^n = \mathrm{Span}\{e_1, e_2, \ldots, e_n\}$

$$e_1 = \begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix} \quad e_2 = \begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix} \quad \cdots \quad e_n = \begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$$

defining condition

Eg: $V = \{(x,y,z) : \boxed{x+y=z}\}$

The defining condition tells you if $(x,y,z)$ is in $V$ or not.

(1) We have to show that if $u=(x_1,y_1,z_1)\in V$ and $v=(x_2,y_2,z_2)\in V$ then their sum is in $V$.

Know: $x_1+y_1=z_1$ $\quad$ $x_2+y_2=z_2$

defining conditions for $u$ & $v$

$u+v = (x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

Want: $(x_1+x_2)+(y_1+y_2)=(z_1+z_2)$ ✓

defining condition for $u+v$.

Since $u+v$ satisfies the defining condition, $u+v\in V$.

(2) We have to show that if $(x,y,z)\in V$ and $c\in\mathbb{R}$ then $c(x,y,z)=(cx,cy,cz)\in V$.

Know: $x+y=z$ $\quad$ Want: $cx+cy=cz$ ✓

Since $cu$ satisfies the defining condition, $cu\in V$.

(3) Is $\begin{pmatrix}0\\0\\0\end{pmatrix}\in V$? Does it satisfy the defining condition?

$0+0=0$ ✓

Since $V$ satisfies the 3 criteria, it is a subspace. ✓

NB: This means $V$ is a span!
How to find a spanning set?
More on this later.

In order to show that a subset is not a subspace, you just have to produce one counterexample to one of the axioms.

defining condition

Eg: $V = \{(x,y) : x \geq 0, y \geq 0\}$

(2) is false: $(1,1) \in V \quad (1 \geq 0, 1 \geq 0)$

but $(-1)(1,1) \notin V \quad (-1 < 0, -1 < 0)$

In practice you will rarely check that a subset is a subspace by verifying the axioms — but you'll show it's not a subspace by finding a counter-example.

**Fact: A span is a subspace**

Proof: Let $V = \mathrm{Span}\{v_1, \dots, v_n\}$.
Here the defining condition for a vector to be in $V$ is that it is a linear combination of $v_1, \dots, v_n$.

(1) We need to show that if

$$c_1v_1+\cdots+c_nv_n \in V \quad \& \quad d_1v_1+\cdots+d_nv_n \in V$$

then their sum is in V: the sum of two linear combos of $v_1, \ldots, v_n$ is a linear combo.

$$(c_1v_1+\cdots+c_nv_n)+(d_1v_1+\cdots+d_nv_n)$$
$$=(c_1+d_1)v_1+\cdots+(c_n+d_n)v_n \in V \checkmark$$

(2) We need to show that if $c_1v_1+\cdots+c_nv_n \in V$ and $d \in \mathbb{R}$ then the product is in V.

$$d(c_1v_1+\cdots+c_nv_n) = (dc_1)v_1+\cdots+(dc_n)v_n \in V \checkmark$$

(3) Every span contains 0:

$$0 = 0v_1+\cdots+0v_n \checkmark$$

Conversely, suppose V is a subspace.

If $v_1, \ldots, v_n \in V$ and $c_1, \ldots, c_n \in \mathbb{R}$ then:

$c_1v_1, \ldots, c_nv_n \in V$ by (2)

$c_1v_1+c_2v_2 \in V$ by (1)

$(c_1v_1+c_2v_2)+c_3v_3 \in V$ by (1)

$\vdots$

$c_1v_1+\cdots+c_nv_n \in V$

So Span$\{v_1, \ldots, v_n\}$ is contained in V.

Choose enough $v_i$'s to fill up V, and you get:

$$\boxed{\text{Subspaces are spans} \quad \text{and} \quad \text{Spans are subspaces.}}$$

**Def:** The column space of a matrix $A$ is the span of its columns.

**Notation:** $\mathrm{Col}(A) = \mathrm{Span}\{\text{cols of } A\}$

This is a subspace of $\mathbb{R}^m$ $\quad m = \#\text{rows}$
(each column has $m$ entries)

$\rightsquigarrow$ column picture.

Since a column space is a span & a span is a subspace, a column space is a subspace.

**Eg:** $\mathrm{Col}\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} = \mathrm{Span}\left\{\begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\5\\6\end{bmatrix}, \begin{bmatrix}7\\8\\9\end{bmatrix}\right\}$

Spans & Col spaces are interchangeable:

**Eg:** $\mathrm{Span}\left\{\begin{bmatrix}2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\end{bmatrix}\right\} = \mathrm{Col}\begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

**NB:** $\boxed{\mathrm{Col}(A) = \{Ax : x \in \mathbb{R}^n\}}$

because "$Ax$" is just a LC of the cols of $A$.

Translation of the column picture criterion for consistency:

$$\boxed{Ax=b \text{ is consistent} \iff b \in \mathrm{Col}(A)}$$

"$b$ can be written as $Ax \iff b \in \mathrm{Col}(A)$"

**Def:** The null space of a matrix $A$ is the solution set of $Ax=0$.

**Notation:** $\mathrm{Nul}(A) = \{x \in \mathbb{R}^n : Ax=0\}$

This is a subspace of $\mathbb{R}^n$ $\quad n = \#$ columns
($n = \#$ variables and $\mathrm{Nul}(A)$ is a solution set)

$\rightsquigarrow$ row picture

**Fact:** $\mathrm{Nul}(A)$ is a subspace

Of course we also know $\mathrm{Nul}(A)$ is a span, but we can verify this directly.

**Proof:** The defining condition for $v \in \mathrm{Nul}(A)$ is that $Av=0$.

(1) Say $u, v \in \mathrm{Nul}(A)$. Is $u+v \in \mathrm{Nul}(A)$?

$A(u+v) = Au + Av = 0 + 0 = 0$ ✓

(2) Say $u \in \text{Nul}(A)$ and $c \in \mathbb{R}$.
Is $cu \in \text{Nul}(A)$?

$$A(cu) = c(Au) = c \cdot 0 = 0 \checkmark$$

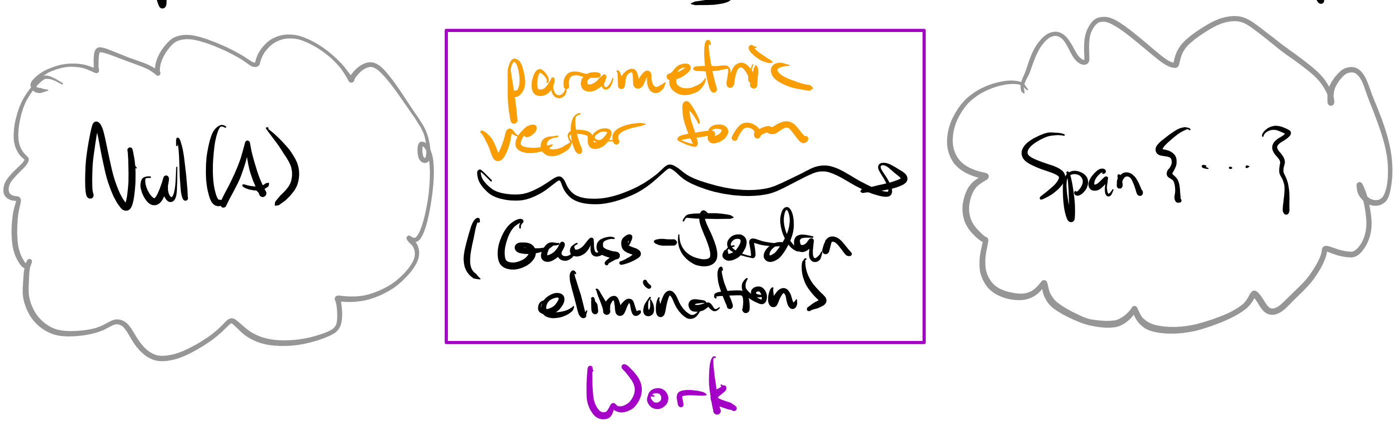
(3) Is $0 \in \text{Nul}(A)$?

$$A0 = 0 \checkmark$$

This is an example of a subspace that does not come with a spanning set!

$\rightsquigarrow$ It's much more natural to consider it as a subspace when reasoning about it.

How to produce a spanning set for a null space?




Eg: Write Nul(A) as a span for

$$A = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix}$$

This means solving $Ax = 0$ (homogeneous equation).

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 1 & 0 \\ 2 & 4 & 1 & -1 & 0 \end{array}\right] \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{form}]{\text{parametric}} \begin{cases} x_1 = -2x_2 + x_4 \\ x_2 = \quad x_2 \\ x_3 = \quad -x_4 \\ x_4 = \quad x_4 \end{cases}$$

$$\xrightarrow{\text{PVF}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\Longrightarrow \mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$$

NB: Any two non-collinear vectors span a plane, so $\mathrm{Nul}(A)$ will have many different spanning sets.

eg $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}}_{\text{sum}}, \underbrace{\begin{pmatrix} -3 \\ 1 \\ 1 \\ -1 \end{pmatrix}}_{\text{difference}} \right\}$

More on this later.

NB: Likewise for the column space: eg.

$$\mathrm{Col}\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Col}\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 0 & 0 \end{pmatrix} = \mathrm{Col}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \{xy\text{-plane}\}$$

# Implicit vs Parametric form:

- Col(A) is a span:

$$\mathrm{Col}\begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} = \text{vectors of the form } \overbrace{x_1}^{\text{parameters}}\begin{pmatrix}1\\2\\3\end{pmatrix} + x_2\begin{pmatrix}4\\5\\6\end{pmatrix} + x_3\begin{pmatrix}7\\8\\9\end{pmatrix}$$

$\rightsquigarrow$ parametric form

- Nul(A) is a solution set:

$$\mathrm{Nul}\begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 4 & 1 & -1 \end{bmatrix} = \left\{ (x_1, x_2, x_3, x_4) : \overbrace{\begin{matrix} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 4x_2 + x_3 - x_4 = 0 \end{matrix}}^{\text{equations}} \right\}$$

$\rightsquigarrow$ implicit form

In practice you will (almost) always write a subspace as a column space/span or a null space. Which one?

- parameters? $\rightsquigarrow$ Col(A) / Span
- equations? $\rightsquigarrow$ Nul(A)

Once you're done this, you can ask a computer to do computations on it!

Eg: $V = \{(x,y,z) : x+y=z\}$

This is defined by the equation $x+y=z$.

rewrite: $x+y-z=0$

$$\rightsquigarrow V = \mathrm{Nul}\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}$$

Eg: $V = \left\{ \begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} : a, b \in \mathbb{R} \right\}$

This is described by parameters. Rewrite:

$$\begin{pmatrix} 3a+b \\ a-b \\ b \end{pmatrix} = a\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\rightsquigarrow V = \mathrm{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\} = \mathrm{Col}\begin{bmatrix} 3 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$$

This is also how you should verify that a subset is a subspace.

Of course, if $V$ is not a subspace then you can't write it as $\mathrm{Col}(A)$ or $\mathrm{Nul}(A)$. In this case you should check that it fails one of the axioms.

Eg: Is $V = \{(x,y,z) : x+y=z+1\}$ a subspace?

No, (P3) fails: $0+0 \neq 0+1$, so $0 \notin V$.